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# SCHRÖDINGER'S CAT

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ABSTRACT

We show that the classical framework of probability spaces, which does not admit a model-theoretical treatment, is equivalent to that of probability algebras, which does. We prove that the category of probability algebras is a stable cat, where non-dividing coincides with the ordinary notion of independence used in probability theory.

## Introduction

In this paper we wish to present a model-theoretic treatment of probability spaces. The objects we are interested in are events and random variables, and their types are going to be their probabilities, or their distributions. And of course, these objects admit the natural notion of probabilistic independence: as we would hope, it turns out to coincide with non-dividing, and is in fact the unique stable (or simple) notion of independence that the "theory" of probability spaces admits.

In doing so there are two main hurdles to be passed. First, let us recall that a **probability space** is classically defined as a triplet $(\Omega, \mathfrak{B}, \mu)$ where $\Omega$ is a set, $\mathfrak{B} \subseteq \mathbb{P}(\Omega)$ is a $\sigma$-algebra of subsets of $\Omega$, and $\mu\colon \mathfrak{B} \to [0,1]$ is a $\sigma$-additive positive measure of total mass 1. The elements of $\mathfrak{B}$ are called **events**, and if $a \in \mathfrak{B}$ then its measure $\mu(a)$ is also called its **probability**. This point-oriented

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description fits our intuition of what a measure space or a measurable function is. On the other hand, the class of probability spaces as two-sorted structures in the signature $(\in, \mu)$ does not seem to have the "nice" properties which would allow any reasonable model-theoretic treatment (as we understand it).

The problematic part of this two-sorted structure seems to be the sort of points: For example, given events $\{a_i : i < \omega\}$, the property $a = \bigcup_{i<\omega} a_i$, which is of fundamental importance to measure theory, cannot conceivably be defined by a (finitary!) formula with parameters in $\{a_i\}$, or even an infinite conjunction thereof, as long as we insist on seeing events as sets of points. But then again, we are principally interested in events and random variables, not in points, so we look for an alternative approach that would forgo points entirely. Forgetting the sort of points, we are left with the boolean algebra $\mathfrak{B}$ of events equipped with a measure function $\mu \colon \mathfrak{B} \to [0,1]$. In the terminology of [Fre04], $\mathfrak{B}$ is a **measure algebra**; as we are mostly interested in the case where the total measure is 1, we study in this paper the model theory of **probability measure algebras**.

The framework of (probability) measure algebras turns out to suffice for all practical purposes, with some definitions somehow turned around. For example, random variables are now defined by the events that in the classical approach they define (e.g., if $f$ is a positive random variable, then we identify it with the sequence of events $(\{f > t\} : t \in \mathbb{Q}^+))$. The pure boolean algebra $\mathfrak{B}$ does not come with a notion of countable union; however, the measure $\mu$ induces one such notion, which is the unique one with respect to which $\mu$ is $\sigma$-additive: thus, in this approach, $\sigma$-additivity comes for free. Also, the existence of "countable unions" in $\mathfrak{B}$ (i.e., the analogue of $\mathfrak{B}$ being a $\sigma$-algebra) is equivalent to $\mathfrak{B}$ being complete in a natural metric; if it is not then it has a unique completion, so in some sense being a $\sigma$-algebra comes for free as well.

We will try to give a pretty complete though schematic introduction to integration theory in probability measure algebras. For a more general treatment, we refer the reader to [Fre04].

The second hurdle is that probability measure algebras do not admit a *first order* treatment. The author does not consider this to be much of a deficiency, as they do admit a natural and elegant treatment as a *compact abstract theory (cat)*. As such it is stable, and in fact $\omega$-stable. It is not uncountably categorical; it is $\omega$-categorical though, and has relatively few models in uncountable cardinals (these last properties follow from Maharam's structure theorem [Fre04, 332B] for measure algebras, discussed in Section 2.3).

We refer the reader to [Bena] for a survey of the framework of compact abstract theories. In particular, the reader is referred there for the definition of cats, types, the topology of the type spaces, and the relations between type spaces as given by the type-space functor. We will mostly be concerned with *Hausdorff* cats, i.e., cats whose type spaces are Hausdorff.

Lowercase letters $a, b, \dots$ denote single elements, or tuples thereof: here these will usually be events in a probability measure algebra, but may also be hyperimaginary elements (i.e., quotients of possibly infinite tuples of "real" elements by type-definable equivalence relations), and thus in particular possibly infinite tuples of elements. The precise meaning of an "element" will always be clear from the context or explicitly stated. Uppercase letters $A, B, \dots$ denote sets, i.e., possibly infinite tuples with no fixed enumeration. Script letters $\mathscr{A}, \mathscr{B}, \dots$ denote probability measure algebras. The notation $a \equiv_A a'$ means that $\mathrm{tp}(a/A) = \mathrm{tp}(a'/A)$.

If $a$, $b$ are any two tuples (possibly hyperimaginary) in a universal domain then $b$ is **definable** (**bounded**) over $a$, in symbols $b \in \mathrm{dcl}(a)$ ($b \in \mathrm{bdd}(a)$) if $\mathrm{tp}(b/a)$ has a unique realisation (bounded number of realisations). If $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$ then they are **interdefinable**. If $\mathrm{dcl}(a) = \mathrm{bdd}(a)$ then $a$ is **boundedly closed**.

Sometimes we wish to view $\mathrm{dcl}(a)$ ($\mathrm{bdd}(a)$) as a set, rather than a proper class. For this we may restrict it to all *small* hyperimaginary elements (i.e., quotients of tuples which are not longer than the cardinality of the language) which are definable (bounded) over $a$: since every hyperimaginary element is interdefinable with a tuple of small ones, there is no loss of information.

## 1. Point-free probability spaces

Here we develop the notion of a probability measure algebra (or probability algebra, for short), explain why it is equivalent to classical probability spaces, and sketch the development of integration theory in this framework.

### 1.1. Probability algebras.

*Definition 1.1:* Let $\mathscr{A}$ be a boolean algebra. A **probability semi-measure** on $\mathscr{A}$ is a function $\mu\colon \mathscr{A} \to [0,1]$ satisfying:

(i) $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$,

(ii) $\mu(1) = 1$,

(iii) $\mu(0) = 0$.

It is a **probability measure** if in addition:

(iii)′ $\mu(a) = 0 \Longleftrightarrow a = 0$.

If $\mu$ is a probability (semi-)measure on $\mathscr{A}$ then we say that $(\mathscr{A}, \mu)$ is a **probability (semi-)measure algebra**. Most of the time, though, we omit $\mu$ and consider it a part of the structure on $\mathscr{A}$. We may shorten probability measure algebra into probability algebra.

A probability algebra as defined here is the same as a measure algebra in the sense of [Fre04] with total measure one. Since we deal here with finite total measure, some technical conditions introduced in [Fre04] are not required.

*Notation 1.2:* If $\mathscr{A}$ is a boolean algebra, then $\oplus$ denotes symmetric difference in $\mathscr{A}$.

We use $\oplus$ to denote symmetric difference, as we think of it more as the addition of the corresponding boolean ring (see Fact 2.1 below). Throughout this paper we will use $\oplus$ to denote addition in boolean rings and $+$ to denote addition in ordinary characteristic zero rings.

*Definition 1.3:* If $\mathscr{A}$ is a probability semi-measure algebra, we define a semi-metric on $\mathscr{A}$ by $d(a,b) = \mu(a \oplus b)$. If $\mu$ is a measure, then $d$ is a metric.

Given a semi-metric space, it is natural to replace it with a metric one, and then with its completion. Both procedures are compatible with the measure algebra structure:

First we define $a \equiv_0 b$ if $d(a,b) = 0$. Then $\equiv_0$ is the equivalence relation induced by the null-measure ideal in $\mathscr{A}$, so $\mathscr{A}/{\equiv_0}$ is a boolean algebra on which $\mu$ is well-defined. Thus $(\mathscr{A}/{\equiv_0}, \mu)$ is a probability measure algebra on which $d$ is a true metric.

Assume now that $\mathscr{A}$ is probability measure algebra. Then it is easy to verify that its unique completion (as a metric space) $\hat{\mathscr{A}}$ admits a unique structure of a probability measure algebra: if $(a_i)$ and $(b_i)$ are Cauchy sequences in $\mathscr{A}$, then $(\lim a_i) \wedge (\lim b_i) = \lim(a_i \wedge b_i)$, $\mu(\lim a_i) = \lim \mu(a_i)$, etc.

In particular, since the metric $d$ was defined from $\mu$, the measure function $\mu$ is always continuous.

*Definition 1.4:* Let $\mathscr{A}$ be a probability algebra, and $\{a_i \colon i < \lambda\} \subseteq \mathscr{A}$ a family of events. We say that $b \in \mathscr{A}$ is the **least upper bound** of $\{a_i \colon i \leq \lambda\}$, in symbols $b = \bigvee_{i<\lambda} a_i$, if $b \geq a_i$ for all $i < \lambda$ and $\mu(b) = \sup_{w \in [\lambda]^{<\omega}} \mu(\bigvee_{i \in w} a_i)$. In case $\lambda = \omega$, which is the interesting one, then this is the same as requiring that $\mu(b) = \lim_{n<\omega} \mu(\bigvee_{i<n} a_i)$.

Greatest lower bounds are defined similarly.

FACT 1.5: Let $\{a_i: i < \lambda\}$ be a family of events in a probability algebra $\mathscr{A}$.

(i) If $b = \bigvee_{i<\lambda} a_i$ (in the sense of probability algebras) then $b$ is the l.u.b. of that family in the sense of pure boolean algebras as well. It is therefore unique.

(ii) There exists a countable subset $I \subseteq \lambda$ (depending on the family $\{a_i: i < \lambda\}$) such that $\bigvee_{i<\lambda} a_i$ exists if and only if $\bigvee_{i\in I} a_i$ does, in which case they are equal.

(iii) If $\lambda = \omega$, then $\bigvee_{i<\omega} a_i = \lim_{n<\omega} \bigvee_{i<n} a_i$ provided at least one exists.

*Proof:*

(i) Assume that $b = \bigvee_{i<\lambda} a_i$, and $c \geq a_i$ for all $i$. Then

$$\mu(b) = \sup_{w\in[\lambda]^{<\omega}} \mu(\bigvee_{i\in w} a_i) \Longrightarrow \mu(b \setminus c) = 0 \Longrightarrow c \geq b.$$

(ii) Let $w_n \in [\lambda]^{<\omega}$ for $n < \omega$ be such that

$$\sup_{n<\omega} \mu(\bigvee_{i\in w_n} a_i) = \sup_{w\in[\lambda]^{<\omega}} \mu(\bigvee_{i\in w} a_i),$$

and let $I = \bigcup_n w_n$. Then

(1)
$$\sup_{w\in[I]^{<\omega}} \mu(\bigvee_{i\in w} a_i) = \sup_{w\in[\lambda]^{<\omega}} \mu(\bigvee_{i\in w} a_i).$$

Clearly, if $b = \bigvee_{i<\lambda} a_i$ exists then $\mu(b)$ is equal to the common value in (1), and $b = \bigvee_{i\in I} a_i$ as well. Conversely, assume $b = \bigvee_{i\in I} a_i$ exists. Then again $\mu(b)$ is equal to the common value in (1). For every $j < \lambda$ we must have $\mu(a_j \setminus b) = 0$, since the contrary would yield a contradiction to (1). Therefore $b \geq a_j$ for all $j < \lambda$ and $b = \bigvee_{i<\lambda} a_j$.

(iii) Immediate from the definitions. $\blacksquare_{1.5}$

More generally, one can show that $\bigvee_{i<\lambda} a_i = \lim_{w\in[\lambda]^{<\omega}} \bigvee_{i\in w} a_i$ provided that at least one exists, when the limit is taken over the directed set of finite subsets of $\lambda$.

By passing to the complement, a probability measure algebra admits least upper bounds if and only if it admits greatest lower bounds. This is further equivalent to completeness:

PROPOSITION 1.6: *A probability measure algebra $\mathscr{A}$ is complete if and only if it admits countable (arbitrary) least upper bounds.*

*Proof:* Assume that $\mathscr{A}$ is complete. Since the measure is finite, a sequence of the form $(\bigvee_{i<n} a_i \colon n < \omega)$ is a Cauchy sequence, and therefore its limit exists, and equal to $\bigvee_{i<\omega} a_i$. By Fact 1.5, if every countable family has a least upper bound, so does in fact every family.

Conversely, assume that $\mathscr{A}$ admits countable least upper bounds, and therefore greatest lower bounds as well. We can define for an arbitrary sequence $(a_i)$

$$\limsup a_i = \bigwedge_{n<\omega} \bigvee_{n\leq i<\omega} a_i \quad \text{and} \quad \liminf a_i = \bigvee_{n<\omega} \bigwedge_{n\leq i<\omega} a_i.$$

Now let $(a_i)$ be a Cauchy sequence. If it has a converging sub-sequence then it converges as well, so we may assume that $d(a_i, a_{i+1}) < 2^{-i}$. It follows that $\max\{d(a_n, \bigwedge_{n\leq i<\omega} a_i), d(a_n, \bigvee_{n\leq i<\omega} a_i)\} < 2^{1-n}$, so $\limsup a_i = \liminf a_i$, and the sequence converges to this common value. $\blacksquare_{1.6}$

Thus a complete probability algebra is the analogue of a $\sigma$-algebra equipped with a $\sigma$-additive probability measure.

*Remark 1.7:* One could try to define an abstract $\sigma$-algebra as a pure boolean algebra where every countable family has a lowest upper bound. However, we do not see how this could work out:

(i) As we said above, without measure we can say that $b \geq a_i$ for all $i$, but using finitary logic we cannot say that $b$ is minimal as such. One undesirable consequence is that if $\mathscr{A}$ is a pure boolean algebra and $b = \bigvee_{i\in I} a_i$ in the sense of $\mathscr{A}$, but $b$ cannot be expressed as a finite sub-disjunction, then there is a boolean algebra $\mathscr{A} \leq \mathscr{A}'$ and $c \in \mathscr{A}'$ such that $c = \bigvee_{i\in I} a_i$ in the sense of $\mathscr{A}'$, and $c \neq b$. Thus, even though $\bigvee_{i\in I} a_i$ may have a value in one pure boolean algebra, it is meaningless in the *category of boolean algebras.*

(ii) If one tries to prove that an abstract $\sigma$-algebra thus defined is equal to a $\sigma$-algebra of *sets*, one may run into difficulties, and in fact we do not know if this is in general true. On the other hand, we prove below that a probability algebra is equivalent to a concrete probability space, *modulo null-measure sets.*

We find this reason enough to claim that in the abstract setting, the measure is an essential part of the structure of a $\sigma$-algebra, not to mention that the measure allows us to remove the "$\sigma$-" prefix without any loss.

The following is more or less evident:

PROPOSITION 1.8: *Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space. Then $(\mathfrak{B}, \mu)$ is a probability semi-measure algebra, and $(\mathfrak{B}/{\equiv_0}, \mu)$ is a complete probability algebra.*

The converse is somewhat more tricky: we know that $(\mathfrak{B}/{\equiv_0}, \mu)$ does not determine $(\Omega, \mathfrak{B}, \mu)$, just as a boolean algebra can have several distinct representations as a boolean algebra of sets.

Recall that the **Stone space** of a boolean algebra $\mathscr{A}$, which will be denoted here by $\tilde{\mathscr{A}}$, is the set of all ultrafilters on $\mathscr{A}$. For every $a \in \mathscr{A}$ we define $\tilde{a} = \{u \in \tilde{\mathscr{A}}: a \in u\} \subseteq \tilde{\mathscr{A}}$: the family of sets $\{\tilde{a}: a \in \mathscr{A}\}$ forms a clopen basis for a totally disconnected topology on $\tilde{\mathscr{A}}$. In particular, $\mathscr{A}$ has a canonical representation as an algebra of sets, namely as the algebra of clopen sets in $\tilde{\mathscr{A}}$.

This suggests that if $(\mathscr{A}, \mu)$ is a probability algebra, we might try and present it as $(\mathfrak{B}(\tilde{\mathscr{A}})/{\equiv_0}, \tilde{\mu})$, where $\mathfrak{B}(\tilde{\mathscr{A}})$ is the algebra of Borel sets in $\tilde{\mathscr{A}}$ and $\tilde{\mu}$ is a probability measure on it. Of course, $(\mathfrak{B}(\tilde{\mathscr{A}})/{\equiv_0}, \tilde{\mu})$ would necessarily be complete, but up to this limitation everything works fine:

THEOREM 1.9: *Let $(\mathscr{A}, \mu)$ be a probability algebra (not necessarily complete). Let $\tilde{\mathscr{A}}$ denote the Stone space of $\mathscr{A}$, and let $\mathfrak{B}(\tilde{\mathscr{A}})$ denote the family of Borel sets in $\tilde{\mathscr{A}}$; for $a \in \mathscr{A}$, let $\tilde{a} \subseteq \tilde{\mathscr{A}}$ be the corresponding clopen set.*

(i) *There is a unique Borel measure $\tilde{\mu}$ on $\tilde{\mathscr{A}}$ such that for every $a \in \mathscr{A}$,*

$$\tilde{\mu}(\tilde{a}) = \mu(a). \tag{2}$$

*And for every $A \in \mathfrak{B}(\tilde{\mathscr{A}})$,*

$$\begin{aligned} \tilde{\mu}(A) &= \sup\{\tilde{\mu}(F): \text{closed } F \subseteq A\} \\ &= \inf\{\tilde{\mu}(U): \text{open } U \supseteq A\}. \end{aligned} \tag{3}$$

*Moreover, if $U$ is open then $\tilde{\mu}(U) = \sup\{\mu(a): \tilde{a} \subseteq U\}$, and if $F$ is closed then $\tilde{\mu}(F) = \inf\{\mu(a): \tilde{a} \supseteq F\}$.*

*In particular, $(\tilde{\mathscr{A}}, \mathfrak{B}(\tilde{\mathscr{A}}), \tilde{\mu})$ is a probability space.*

(ii) *If $(\mathscr{A}, \mu)$ is complete, then every Borel set (or for that matter, every Lebesgue-measurable set) $A \subseteq \tilde{\mathscr{A}}$ is $\equiv_0$-equivalent to a unique clopen set $\tilde{a}_A \subseteq \tilde{\mathscr{A}}$. In particular, identifying $a \in \mathscr{A}$ with $\tilde{a}/{\equiv_0}$, we have $(\mathfrak{B}(\tilde{\mathscr{A}})/{\equiv_0}, \tilde{\mu}) = (\mathscr{A}, \mu)$.*

*Moreover, if $U \subseteq \tilde{\mathscr{A}}$ is open then $U \subseteq \tilde{a}_U$, and if $F \subseteq \tilde{\mathscr{A}}$ is closed then $F \supseteq \tilde{a}_F$.*

(iii) *If $(\mathscr{A}, \mu)$ is not complete then $(\mathfrak{B}(\tilde{\mathscr{A}})/{\equiv_0}, \tilde{\mu})$ is its completion, namely the minimal complete probability algebra extending $(\mathscr{A}, \mu)$.*

*Proof:*

(i) Define an appropriate Riemann integral $\int f d\mu$ of complex-valued functions $f\colon \tilde{\mathscr{A}} \to \mathbb{C}$ using finite partitions of $\tilde{\mathscr{A}}$ into clopen sets: since the clopen sets are precisely those of the form $\tilde{a}$ for $a \in \mathscr{A}$, they have an associated measure. Restricted to continuous functions, the Riemann integral always exists (just as in the case of functions from finite intervals of the reals) and forms a positive functional. By the Riesz representation theorem [Rud66, Theorem 2.14] there is a (unique) Borel measure $\tilde{\mu}$ on $\tilde{\mathscr{A}}$ such that $\int f d\mu = \int f d\tilde{\mu}$ for every continuous $f$, which in addition satisfies (3) (since $\tilde{\mathscr{A}}$ is compact and the total measure is finite). As characteristic functions of clopen sets are continuous, we immediately obtain (2). This shows that $\tilde{\mu}$ exists.

We know that in a Stone space, if $F \subseteq U$ are closed and open, respectively, then there is $a \in \mathscr{A}$ such that $F \subseteq \tilde{a} \subseteq U$. Using this fact, the moreover part follows from (3); along with (2) and (3) it determines $\tilde{\mu}$ entirely, whence the uniqueness.

(ii) We assume now that $(\mathscr{A}, \mu)$ is complete. We need to prove that for every $A \in \mathfrak{B}(\tilde{\mathscr{A}})$ there exists $a_A \in \mathscr{A}$ such that $A \equiv_0 \tilde{a}_A$: if one exists then it is unique.

Consider first the case of $U \subseteq \tilde{\mathscr{A}}$ open, and let $a_U = \bigvee_{\tilde{a} \subseteq U} a$. Then $U \subseteq \tilde{a}_U$, and $\mu(a_U) = \sup\{\mu(a)\colon \tilde{a} \subseteq U\} = \tilde{\mu}(U)$, whereby $U \equiv_0 \tilde{a}_U$. This also yields the moreover part (for closed sets, pass to complement).

We now proceed by induction on the construction of Borel sets. If $A \equiv_0 \tilde{a}_A$ then $\tilde{\mathscr{A}} \smallsetminus A \equiv_0 \tilde{a}^c_A$, so $a_{\tilde{\mathscr{A}} \smallsetminus A} = a^c_A$. If $A_i \equiv_0 \tilde{a}_{A_i}$ for every $i < \omega$, then $\bigcup A_i \equiv_0 \bigcup \tilde{a}_{A_i} \equiv_0 \widetilde{\bigvee a_{A_i}}$, so $a_{\bigcup A_i} = \bigvee a_{A_i}$. This shows that all Borel sets are clopen modulo $\equiv_0$.

(iii) Finally, assume that $(\mathscr{A}, \mu) \subseteq (\mathscr{A}', \mu')$ where $(\mathscr{A}', \mu')$ is complete, but $(\mathscr{A}, \mu)$ maybe not so. Then the inclusion $\mathscr{A} \hookrightarrow \mathscr{A}'$ induces a projection $\tilde{\mathscr{A}}' \twoheadrightarrow \tilde{\mathscr{A}}$ which in turn induces an inclusion $\mathfrak{B}(\tilde{\mathscr{A}}) \hookrightarrow \mathfrak{B}(\tilde{\mathscr{A}}')$. From $\mu \subseteq \mu'$ we easily get $\tilde{\mu} \subseteq \tilde{\mu}'$, whereby $(\mathfrak{B}(\tilde{\mathscr{A}})/{\equiv_0}, \tilde{\mu}) \subseteq (\mathfrak{B}(\tilde{\mathscr{A}}')/{\equiv_0}, \tilde{\mu}') = (\mathscr{A}', \mu')$. $\blacksquare_{1.9}$

Compare this now with the point-oriented approach: if $\mu$ is additive, but not $\sigma$-additive, on some $\sigma$-algebra $\mathfrak{B}$, say $\mu(\bigcup_{i<\omega} A_i) > \lim \mu(\bigcup_{i<n} A_i)$, then this just means that there are missing points whose measure should have been the difference. Thus, points are not only unnecessary, they can be simply misleading.

When $(\mathscr{A}, \mu)$ is complete, the measure space $(\tilde{\mathscr{A}}, \mathfrak{B}(\tilde{\mathscr{A}}), \tilde{\mu})$ is particularly nice.

COROLLARY 1.10: *Assume that $(\mathscr{A}, \mu)$ is complete.*

(i) *For every Borel set $A$,*

$$\tilde{\mu}(A) = \sup\{\mu(a)\colon \tilde{a} \subseteq A\} = \tilde{\mu}(\mathring{A})$$
$$= \inf\{\mu(a)\colon \tilde{a} \supseteq A\} = \tilde{\mu}(\bar{A}).$$

(ii) *The meager sets ideal and the zero measure sets ideal coincide in $\tilde{\mathscr{A}}$. In fact, a set is meager if and only if it is nowhere-dense.*

*Proof:*

(i) We know that for open $U \subseteq \tilde{\mathscr{A}}$, $U \subseteq \tilde{a}_U$ and $\tilde{\mu}(U) = \mu(a_U)$. Therefore,

$$\begin{aligned}\tilde{\mu}(A) &\leq \inf\{\mu(a)\colon \tilde{a} \supseteq A\}\\ &\leq \inf\{\mu(a_U)\colon U \text{ is open and } \tilde{a}_U \supseteq A\}\\ &\leq \inf\{\mu(a_U)\colon U \text{ is open and } U \supseteq A\}\\ &= \inf\{\tilde{\mu}(U)\colon U \text{ is open and } U \supseteq A\}\\ &= \tilde{\mu}(A).\end{aligned}$$

And $\tilde{\mu}(A) = \sup\{\mu(a)\colon \tilde{a} \subseteq A\}$ is obtained by passing to the complement.

The two other equalities are obtained from these and $\tilde{a} \subseteq \mathring{A} \iff \tilde{a} \subseteq A$, $\tilde{a} \supseteq \bar{A} \iff \tilde{a} \supseteq A$.

(ii) Let $A$ be a Borel set. Then $\tilde{\mu}(A) = 0 \iff \tilde{\mu}(\bar{A}) = 0 \iff \tilde{\mu}(\mathring{\bar{A}}) = 0 \iff \mathring{\bar{A}} = \varnothing$. Thus a set has zero measure if and only if it is nowhere-dense. Since the zero measure sets form a $\sigma$-ideal, so do the nowhere-dense ones in this case. In general, the family of meager sets is the $\sigma$-ideal generated by the nowhere-dense sets, so here the two notions agree. $\blacksquare_{1.10}$

*Definition 1.11:* Let $\mathfrak{M}$ denote the category of probability algebras, where morphisms are morphisms of boolean algebras, which in addition preserve the measure.

Thus the category $\mathfrak{M}$ considers the measure a part of the information contained in a probability algebra $\mathscr{A} \in \mathfrak{M}$, so we omit $\mu$ from the notation. If we want to be explicit about it, we may still write $\mu_{\mathscr{A}}$ for the measure of $\mathscr{A}$.

We write $\mathscr{A} \leq \mathscr{B}$ to say that $\mathscr{A} \subseteq \mathscr{B}$ and the inclusion is a morphism.

1.2. MEASURABLE FUNCTIONS. We recall that a **positive measurable function** defined on a probability space $(\Omega, \mathfrak{B}, \mu)$ is a function $f\colon \Omega \to [0, \infty]$ such that for every $t \in [0, \infty]$ the set $\{x\colon f(x) > t\}$ is measurable. We may denote this set simply by $\{f > t\}$. Clearly, it suffices to require this for every $t \in \mathbb{Q}^+$. Two functions are said to be equal **almost everywhere (a.e.)** if they differ on a zero measure set.

The analogous definition is:

*Definition 1.12:* A **positive function** on a complete probability algebra $\mathscr{A}$ is a decreasing sequence $f = (f_t: t \in \mathbb{Q}^+)$ satisfying $f_t = \bigvee_{s>t} f_s$. We write (with some abuse) $f: \mathscr{A} \to [0, \infty]$.

The event $f_0$ is also called the **support** of $f$, also denoted $\operatorname{supp} f$.

If $\mathscr{A}$ is not complete, then a positive function on $\mathscr{A}$ is defined as a positive function on its completion.

Intuitively, we view $f_t$ as the event $\{f > t\}$, whence the definition of support and the requirement that $f_t = \bigvee_{s>t} f_s$.

It is quite reassuring that the two notions agree:

PROPOSITION 1.13: *Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space and $(\mathfrak{B}/{\equiv_0}, \mu)$ the associated probability measure algebra. If $f: \Omega \to [0, \infty]$ is measurable, define $\hat{f}$ as the sequence $(\{f > t\}/{\equiv_0}: t \in \mathbb{Q}^+)$ in $\mathfrak{B}/{\equiv_0}$. Then the map $f \mapsto \hat{f}$ is a bijection between the set of positive measurable functions on $\Omega$ modulo equality a.e. and the set of positive functions on $\mathfrak{B}/{\equiv_0}$.*

*Proof:* If $f, g: \Omega \to [0, \infty]$ are equal a.e. then $\{f > t\} \equiv_0 \{g > t\}$ for every $t$, whereby $\hat{f} = \hat{g}$. Thus the map $f \mapsto \hat{f}$ is well-defined even if $f$ is only known modulo equality a.e. Since $r > t \iff (\exists s > t)(r > s)$, $\hat{f}$ is a positive function on $\mathfrak{B}/{\equiv_0}$.

Conversely, assume that a positive function $g = (g_t: t \in \mathbb{Q}^+)$ is given on $\mathfrak{B}/{\equiv_0}$. Every $g_t$ is of the form $A_t/{\equiv_0}$ for some $A_t \in \mathfrak{B}$. Define $f(x) = \sup\{t: x \in A_t\}$ (we convene that in this context, $\sup \varnothing = 0$). Then $f(x) > t \iff (\exists s > t)(x \in A_s)$, so $\{f > t\} = \bigcup_{s>t} A_s$ and $\{f > t\}/{\equiv_0} = (\bigcup_{s>t} A_s)/{\equiv_0} = \bigvee_{s>t} g_s = g_t$. Thus $\hat{f} = g$, and the map is surjective.

Finally, assume that $f$ and $g$ are not equal a.e. Then $\mu(\{f \neq g\}) > 0$, and we may assume that $\mu(\{f > g\}) > 0$. Since $\{f > g\} = \bigcup_{t \in \mathbb{Q}^+} \{f > t\} \cap \{g < t\}$ and this is a countable union, there is $t$ such that $\mu(\{f > t\} \cap \{g < t\}) > 0$. Therefore $\{f > t\} \not\equiv_0 \{g > t\}$ and $\hat{f} \neq \hat{g}$. ■$_{1.13}$

As before, the situation is even nicer when the probability space under consideration is the Stone space of a complete probability algebra:

PROPOSITION 1.14: *If $\mathscr{A}$ is a complete probability algebra then every positive measurable function on $\tilde{\mathscr{A}}$ is equal a.e. to a unique continuous function $f: \tilde{\mathscr{A}} \to$*

$[0, \infty]$. *Thus, the set of positive functions on* $\mathscr{A}$ *is in bijection with the set of continuous functions from* $\tilde{\mathscr{A}}$ *to* $[0, \infty]$.

*Proof:* It suffices to show that if $g = (g_t)$ is a positive function on $\mathscr{A}$ then there exists a continuous function $f: \tilde{\mathscr{A}} \to [0, \infty]$ satisfying $\hat{f} = g$.

Define $f(x) = \sup\{t: x \in \tilde{g}_t\}$, and we already know from the proof of Proposition 1.13 that $\hat{f} = g$. Moreover, $\{f > t\} = \bigcup_{s>t} \tilde{g}_s$, which is open, and similarly $\{f \geq t\} = \bigcap_{s<t} \tilde{g}_s$, which is closed, so $f$ is continuous. ■1.14

From now on assume that $\mathscr{A}$ is a complete probability algebra.

If $f: \mathscr{A} \to [0, \infty]$ is a positive function on $\mathscr{A}$ and $B \subseteq [0, \infty]$ is a Borel set, then we can define the event $\{f \in B\} \in \mathscr{A}$ by induction on the complexity of $B$:

$$\begin{aligned} \{f \in [0, a]\} = \{f \leq a\} &= \bigwedge_{t>a} f_t^c, \\ \{f \in [a, \infty]\} = \{f \geq a\} &= \bigvee_{t<a} f_t, \\ \{f \in [0, \infty] \setminus B\} &= \{f \in B\}^c, \\ \{f \in \bigcap_{i<\omega} B_i\} &= \bigwedge_i \{f \in B_i\}. \end{aligned}$$

(Since every closed set in $[0, \infty]$ is a countable intersection of finite unions of closed intervals, this is enough.)

If $\alpha \leq \omega$, $f_i$ are positive functions for $i < \alpha$ and $B \subseteq [0, \infty]^\alpha$ is a Borel set, the event $\{\bar{f} \in B\}$ is constructed in a similar manner. If $\varphi: [0, \infty]^\alpha \to [0, \infty]$ is Borel-measurable, then $g = \varphi \circ \bar{f}: \mathscr{A} \to [0, \infty]$ is defined by

$$\{g > t\} = \{\bar{f} \in \varphi^{-1}((t, \infty])\}.$$

For example, if $f, g: \mathscr{A} \to [0, \infty]$ and $p > 0$ we have

$$\begin{aligned} \{f + g > t\} &= \bigvee_{s+r>t} (f_s \wedge g_r), \\ \{fg > t\} &= \bigvee_{sr>t} (f_s \wedge g_r), \\ \{f^p > t\} &= \bigvee_{s^p>t} f_s. \end{aligned}$$

Similarly, if $f_i\colon \mathscr{A} \to [0,\infty]$ for $i < \omega$ then

$$\{\sup f_i > t\} = \bigvee_i \{f_i > t\},$$
$$\{\inf f_i > t\} = \bigvee_{s>t} \bigwedge_i \{f_i > s\},$$
$$\limsup f_i = \inf_n \sup_{i\geq n} f_i,$$
$$\liminf f_i = \sup_n \inf_{i\geq n} f_i.$$

If $\limsup f_i = \liminf f_i$, then $\lim f_i$ is the common value. We leave it as an easy exercise to the reader to verify that these are indeed the notions corresponding to pointwise supremum, infimum and limit of sequences of ordinary measurable functions up to equality a.e.

We may extend the definition of functions to other ranges:

*Definition 1.15:*

(i) A **real-valued positive function** $f\colon \mathscr{A} \to [0,\infty)$ is a positive function satisfying $\bigwedge_t f_t = 0$ (that is to say that $\{f = \infty\}$ is empty).

(ii) A **real-valued function** $f\colon \mathscr{A} \to \mathbb{R}$ is a pair $f = (f^+, f^-)$, where $f^+$ and $f^-$ are positive real-valued functions with disjoint supports. Intuitively, $f = f^+ - f^-$, and $f^+ = f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$.

(iii) A **complex-valued function** $f\colon \mathscr{A} \to \mathbb{C}$ is a pair $f = (u, v)$, where $u$ and $v$ are real-valued functions. Intuitively, $f = u + iv$.

It follows from Proposition 1.13 that if $\mathscr{A}$ is the probability measure algebra of a probability space $(\Omega, \mathfrak{B}, \mu)$, then the set of real-valued positive (real-valued, complex-valued) functions on $\mathscr{A}$ is in bijection with the $[0,\infty)$-valued ($\mathbb{R}$-valued, $\mathbb{C}$-valued) measurable functions on $\Omega$, up to equality a.e.

Let $X \in \{[0,\infty), \mathbb{R}, \mathbb{C}\}$, and $f_i\colon \mathscr{A} \to X$ for $i < \alpha \leq \omega$. Then events of the form $\{\bar{f} \in B\}$ where $B \subseteq X^\alpha$ is a Borel set, as well as the composition $\varphi \circ \bar{f}\colon \mathscr{A} \to X$ for a Borel function $\varphi\colon X^\alpha \to X$ are defined as in the case $X = [0,\infty]$.

For example, for $f, g\colon \mathscr{A} \to [0,\infty)$,

$$\{f \dot{-} g > t\} = \bigvee_{s-r>t} (f_s \setminus g_r) \quad (\text{where } x \dot{-} y = \max\{x-y, 0\}).$$

And for $f, g\colon \mathscr{A} \to \mathbb{R}$,

$$\begin{aligned} f+g &= ((f^+ + g^+) - (f^- + g^-)) - ((f^- + g^-) \dot{-} (f^+ + g^+)), \\ fg &= (f^+g^+ + f^-g^-) - (f^+g^- + f^-g^+), \\ |f| &= f^+ + f^-, \\ &\text{etc.} \end{aligned}$$

Addition, multiplication, absolute value and limits extend to complex-valued functions in a natural manner.

We may now translate to this terminology the basics of the theory of integration given in [Rud66], leaving it to the reader to follow the analogy.

For $a \in \mathscr{A}$, we define its **characteristic function** $\chi_a$ by $\chi_{a,t} = a$ for $0 \leq t < 1$ and $\chi_{a,t} = 0$ for $1 \leq t$. A function of the form $f = \sum_{i<n} \alpha_i \chi_{a_i}$ where $\alpha_i \in [0,\infty]$ and $a_i \in \mathscr{A}$ is called **simple**, and we can always write it such that the $a_i$ be disjoint. If $f = \sum_{i<n} \alpha_i \chi_{a_i}$ is simple we define $\int f = \sum_{i<n} \alpha_i \mu(a_i)$, and this is a value in $[0,\infty]$ which does not depend on the particular representation of $f$ as a simple function. For $f\colon \mathscr{A} \to [0,\infty]$ we define $\int f = \sup_{f \geq g \text{ simple}} \int g$ (here $f \geq g$ if $f_t \geq g_t$ for all $t$, or equivalently, if $g \dot{-} f = 0$).

For a complex-valued function $f\colon \mathscr{A} \to \mathbb{C}$ and $p \in [1,\infty]$ we define

$$\begin{aligned} \|f\|_p &= \left(\int |f|^p\right)^{1/p}, & p \neq \infty, \\ \|f\|_\infty &= \sup\{t \in \mathbb{Q}^+ : f_t \neq 0\}, \\ L_p(\mathscr{A}) &= \{f\colon \mathscr{A} \to \mathbb{C} : \|f\|_p < \infty\}. \end{aligned}$$

Since the total measure is finite, $p < q \implies L_p(\mathscr{A}) \supseteq L_q(\mathscr{A})$, and if $\mathscr{A}$ is infinite then the inclusion is strict. For $f = u + iv \in L_1(\mathscr{A})$ we say that $f$ is **integrable**, and define

$$\int f = \int [(u^+ - u^-) + i(v^+ - v^-)] = \int u^+ - \int u^- + i\int v^+ - i\int v^-.$$

This is a well-defined complex value.

If $a \in \mathscr{A}$ is an event and $f$ is either positive or in $L_1$, then $\int_a f = \int f\chi_a$.

As the definitions we gave coincide with classical definitions when passing to measurable functions on $\tilde{\mathscr{A}}$, we obtain in particular that the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem all hold.

We will also need the following tool:

*Definition 1.16:* Let $\mathscr{A} \leq \mathscr{B}$ be complete probability algebras, and $f\colon \mathscr{B} \to [0,\infty]$ or $f \in L_1(\mathscr{B})$ a function. We say that a function $g\colon \mathscr{A}\colon \to [0,\infty]$ or $g \in L_1(\mathscr{A})$, accordingly, is the **conditional expectation** of $f$ with respect to $\mathscr{A}$, in symbols $g = \mathbb{E}[f\,|\mathscr{A}]$, if for every $a \in \mathscr{A}$,

$$\int_a g = \int_a f. \tag{4}$$

If $b \in \mathscr{B}$ and $g = \mathbb{E}[\chi_b\,|\mathscr{A}]$, we also call $g$ the **conditional probability** of $b$ with respect to $\mathscr{A}$, in symbols $g = \mathbb{P}[b\,|\mathscr{A}]$.

FACT 1.17: Let $\mathscr{A} \leq \mathscr{B}$ be complete probability algebras.

(i) For all $f$ as in Definition 1.16, $\mathbb{E}[f\,|\mathscr{A}]$ exists and is unique. In particular, $\mathbb{P}[b\,|\mathscr{A}]$ exists for all $b \in \mathscr{B}$.

(ii) If $f_0, f_1\colon \mathscr{B} \to [0,\infty]$ and $g_0, g_1\colon \mathscr{A} \to [0,\infty]$, or $f_0, f_1 \in L_p(\mathscr{B})$ and $g_0, g_1 \in L_q(\mathscr{A})$, for some pair of conjugate exponents $p, q$, then

$$\mathbb{E}[g_0 f_0 + g_1 f_1\,|\mathscr{A}] = g_0\mathbb{E}[f_0\,|\mathscr{A}] + g_1\mathbb{E}[f_1\,|\mathscr{A}].$$

(iii) The monotone convergence theorem holds over $\mathscr{A}$, i.e., if $f_n\colon \mathscr{B} \to [0,\infty]$ are increasing and $f = \lim f_n$ pointwise then $\mathbb{E}[f\,|\mathscr{A}] = \lim \mathbb{E}[f_n\,|\mathscr{A}]$. Similarly for Fatou's lemma and dominated convergence.

*Proof:* See [Fre03, Section 233]. $\blacksquare_{1.17}$

## 2. The category of probability algebras

The results of the previous section provide sufficient justification to replace the classic context of probability spaces with that of (complete) probability algebras.

2.1. FREE AMALGAMATION. Recall that a **boolean ring** is a unitary ring where every element is idempotent. This implies in particular that it is commutative of characteristic 2. Moreover:

FACT 2.1: The giving of a boolean algebra structure $(\mathscr{A}, \wedge, \vee, {}^c)$ and of a boolean ring structure $(\mathscr{A}, \cdot, \oplus)$ are equivalent, where the passage in one direction is given by $ab = a \wedge b$, $a \oplus b = (a \vee b) \setminus (a \wedge b)$, and the other by $a \wedge b = ab$, $a \vee b = a \oplus b \oplus ab$, $a^c = 1 \oplus a$.

We therefore identify boolean algebras with the corresponding boolean rings and vice versa.

FACT 2.2: Let $A$ be a commutative unitary ring, and $\mathrm{idem}(A)$ the set of idempotents in $A$. For $a, b \in \mathrm{idem}(A)$ define $a \wedge b = ab$, $a \vee b = a + b - ab$, $a^c = 1 - a$ and $a \oplus b = a + b - 2ab$. Then $(\mathrm{idem}(A), \wedge, \vee, {}^c)$ is a boolean algebra, and $(\mathrm{idem}(A), \cdot, \oplus)$ is the corresponding boolean ring.

*Proof:* One way to see this is to observe that $(\mathrm{idem}(A), \wedge, \vee, {}^c)$ is precisely the boolean algebra of clopen sets in $\mathrm{Spec}(A)$, and that $(\mathrm{idem}(A), \cdot, \oplus)$ is the corresponding boolean ring.

Alternatively, one can verify "by hand" that $(\mathrm{idem}(A), \cdot, \oplus)$ is a ring, since then it is a boolean ring and $(\mathrm{idem}(A), \wedge, \vee, {}^c)$ is the corresponding boolean algebra. Clearly, $0, 1 \in \mathrm{idem}(A)$, and it is closed for products. The rest is verified as follows:

$$\begin{aligned}(a \oplus b)^2 &= (a + b - 2ab)^2 = a^2 + b^2 + 4a^2b^2 + 2ab - 4a^2b - 4ab^2\\ &= a + b - 2ab = a \oplus b,\\ (a \oplus b) \oplus c &= a + b + c - 2ab - 2ac - 2bc + 4abc = a \oplus (b \oplus c),\\ a \oplus 0 &= a + 0 - 0 = a,\\ (a \oplus b)c &= ac + bc - 2abc = ac \oplus bc. \qquad \blacksquare_{2.2}\end{aligned}$$

*Construction 2.3:* Recall that $\mathfrak{M}$ denotes the category of probability measure algebras, whose morphisms preserve the boolean structure as well as the measure (Definition 1.11). Let $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathfrak{M}$, and let $f\colon \mathscr{C} \to \mathscr{A}$, $g\colon \mathscr{C} \to \mathscr{B}$ be morphisms. Identifying them with the corresponding boolean rings, we can define $\mathscr{D} = \mathscr{A} \otimes_{\mathscr{C}} \mathscr{B}$: then $\mathscr{D}$ is also a boolean ring, and we identify it with the corresponding boolean algebra. Define $D = L^\infty(\mathscr{A}) \otimes_{L^\infty(\mathscr{C})} L^\infty(\mathscr{B})$, and consider the sequence

$$\begin{array}{ccccccc} \mathscr{D} & \xrightarrow{\chi'} & \mathrm{idem}(D) & \hookrightarrow & D & \xrightarrow{\mathbb{E}'[-|\mathscr{C}]} & L^\infty(\mathscr{C}) \\ a \otimes b & \longmapsto & \chi_a \otimes \chi_b & & f \otimes g & \longmapsto & \mathbb{E}[f\,|\mathscr{C}]\,\mathbb{E}[g\,|\mathscr{C}] \end{array}$$

where $\chi'$ and $\mathbb{E}'[-|\mathscr{C}]$ are induced by the appropriate $\mathscr{C}$- and $L^\infty(\mathscr{C})$-bilinear maps, respectively, and the inclusion in the middle serves as a means to pass from characteristic 2 to characteristic 0.

For $h \in D$ we can define $\int' h = \int \mathbb{E}'[h\,|\mathscr{C}]$, and for $d \in \mathscr{D}$ define $\mu(d) = \int' \chi'_d$. Then

$$\mu(d \vee d') = \int' (\chi'_d \vee \chi'_{d'}) = \int' (\chi'_d + \chi'_{d'} - \chi'_{dd'}) = \mu(d) + \mu(d') - \mu(dd').$$

It is easy to verify that $\mu(1 \otimes 1) = 1$ and $\mu(0 \otimes 0) = 0$, so $\mu$ is a semi-measure. As every element of $\mathscr{D}$ can be written as the sum of disjoint elements of the form $a \otimes b$, in order to verify that $\mu$ is a measure it would suffice to show that if $\mu(a \otimes b) = 0$ then $a \otimes b = 0$. But $\mu(a \otimes b) = \int \mathbb{E}[\chi_a | \mathscr{A}]\mathbb{E}[\chi_b | \mathscr{C}]$, and this is zero only if $\mathbb{E}[\chi_a | \mathscr{C}]$ and $\mathbb{E}[\chi_b | \mathscr{C}]$ have disjoint supports (since these are positive functions). In this case there is $c \in \mathscr{C}$ (say the support of $\mathbb{E}[\chi_a | \mathscr{C}]$) such that $a \leq f(c)$ and $b \leq g(c)^c$, so $a \otimes b = (a \wedge f(c)) \otimes b = a \otimes (b \wedge g(c)) = 0$ as required.

Define $k\colon \mathscr{A} \to \mathscr{D}$ by $a \mapsto a \otimes 1$. Then $\mu(k(a)) = \mu(a \otimes 1) = \int \mathbb{E}[\chi_a | \mathscr{C}] = \int \chi_a = \mu(a)$, so $k$ is a morphism of probability algebras. Similarly for $l\colon \mathscr{B} \to \mathscr{D}$ defined by $b \mapsto 1 \otimes b$, and moreover $k \circ f = l \circ g$.

We call $\mathscr{D} = \mathscr{A} \otimes_{\mathscr{C}} \mathscr{B}$ (equipped with $k, l$) the **free amalgam** (in the sense of probability algebras) of $\mathscr{A}$ and $\mathscr{B}$ over $\mathscr{C}$.

It should be noted about the difference in approach between model theory and analysis that [Fre04] contains the construction of the free amalgam of two measure algebras over nothing (i.e., over the trivial algebra), whereas we require amalgamation over non-trivial algebras as well.

*Convention 2.4:* From now on, all the probability algebras under consideration are sub-algebras of an ambient probability algebra.

For the time being, until we prove the existence of a universal domain, this ambient algebra may vary and we assume it is part of the context.

*Notation 2.5:* If $\mathscr{A}, \mathscr{B}$ are boolean algebras then $\mathscr{A} \wedge \mathscr{B}$ is the boolean algebra generated (in the ambient algebra) by $\mathscr{A} \cup \mathscr{B}$.

*Definition 2.6:* Assume that $\mathscr{C} \leq \mathscr{A}, \mathscr{B}$ are boolean algebras. We say that $\mathscr{A} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathscr{C}} \mathscr{B}$ if $\mathscr{A} \wedge \mathscr{B} \cong \mathscr{A} \otimes_{\mathscr{C}} \mathscr{B}$.

LEMMA 2.7: *Let $\mathscr{C} \leq \mathscr{A}, \mathscr{B}$. Then the following are equivalent:*

(i) $\mathscr{A} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathscr{C}} \mathscr{B}$.

(ii) *For every $a \in \mathscr{A}$ and $b \in \mathscr{B}$,*

$$\mathbb{P}[a \wedge b | \mathscr{C}] = \mathbb{P}[a | \mathscr{C}]\,\mathbb{P}[b | \mathscr{C}].$$

(iii) *For every two functions $f\colon \mathscr{A} \to [0, \infty]$, $g\colon \mathscr{B} \to [0, \infty]$ or $f \in L_p(\mathscr{A})$, $g \in L_q(\mathscr{B})$ (where $p$ and $q$ are conjugate exponents),*

$$\mathbb{E}[fg | \mathscr{C}] = \mathbb{E}[f | \mathscr{C}]\,\mathbb{E}[g | \mathscr{C}]. \tag{5}$$

*Proof:*

(i)$\Rightarrow$(ii) Since $\mathscr{A} \mathop{\smile\!\!\!\!\!\!\perp}_{\mathscr{C}} \mathscr{B}$, we may identify $\mathscr{A} \wedge \mathscr{B}$ with $\mathscr{A} \otimes_{\mathscr{C}} \mathscr{B}$. Then, with the notation of Construction 2.3, we have for all $a \in \mathscr{A}$, $b \in \mathscr{B}$ and

$c \in \mathscr{C}$,

$$\begin{aligned}\int_c \mathbb{E}[\chi_a \,|\mathscr{C}]\,\mathbb{E}[\chi_b \,|\mathscr{C}] &= \int \chi_c \mathbb{E}[\chi_a \,|\mathscr{C}]\,\mathbb{E}[\chi_b \,|\mathscr{C}] \\ &= \int \mathbb{E}[\chi_a \,|\mathscr{C}]\,\mathbb{E}[\chi_{b\wedge c} \,|\mathscr{C}] \\ &= \int' \chi'_{a\otimes(b\wedge c)} = \mu(a \otimes (b \wedge c)) \\ &= \mu((a\otimes 1) \wedge (1 \otimes (b \wedge c))) \\ &= \mu(a \wedge b \wedge c) = \int_c \chi_{a\wedge b}.\end{aligned}$$

(ii)⇒(iii) Since (5) holds for characteristic functions, by positive linearity it holds for positive simple functions; by the monotone convergence theorem, for arbitrary positive functions; and by $\mathbb{C}$-linearity, for functions whose product is $L_1$.

(iii)⇒(i) The mapping $(a, b) \mapsto a \wedge b$ is $\mathscr{C}$-bilinear, and induces a morphism of boolean algebras $\sigma\colon \mathscr{A} \otimes_{\mathscr{C}} \mathscr{B} \to \mathscr{A} \wedge \mathscr{B}$. It is clearly onto, and

$$\begin{aligned}\mu(a \wedge b) = \int \chi_a\chi_b &= \int \mathbb{E}[\chi_a\chi_b \,|\mathscr{C}] \\ &= \int \mathbb{E}[\chi_a \,|\mathscr{C}]\,\mathbb{E}[\chi_b \,|\mathscr{C}] = \mu(a\otimes b).\end{aligned}$$

Therefore $\sigma$ is an isomorphism of probability algebras. $\blacksquare_{2.7}$

LEMMA 2.8: *Let $\mathscr{A}$ and $\mathscr{C} \leq \mathscr{B}$ be boolean algebras (note that here we do* **not** *require that $\mathscr{C} \leq \mathscr{A}$), and let $\mathscr{A}' = \mathscr{A} \wedge \mathscr{C}$. Then the following are equivalent:*

(i) $\mathscr{A}' \perp_{\mathscr{C}} \mathscr{B}$.

(i) *If $f\colon \mathscr{A} \to [0, \infty]$ or $f \in L_1(\mathscr{B})$, and we consider it as defined on $\mathscr{A} \wedge \mathscr{B}$, then*

$$\mathbb{E}[f \,|\mathscr{C}] = \mathbb{E}[f \,|\mathscr{B}]. \tag{6}$$

*(So, in particular, $\mathbb{E}[f \,|\mathscr{B}]$ is $\mathscr{C}$-measurable.)*

(iii) *(6) holds for characteristic functions (of events in $\mathscr{A}$).*

*Proof:*

(i)⇒(ii) Let $f\colon \mathscr{A} \to [0, \infty]$ or $f \in L_1(\mathscr{A})$. Then we need to verify that for all $b \in \mathscr{B}$,

$$\begin{aligned}\int_b \mathbb{E}[f \,|\mathscr{C}] = \int \mathbb{E}[\chi_b \mathbb{E}[f \,|\mathscr{C}] \,|\mathscr{C}] &= \int \mathbb{E}[f \,|\mathscr{C}]\,\mathbb{E}[\chi_b \,|\mathscr{C}] \\ &= \int \mathbb{E}[f\chi_b \,|\mathscr{C}] = \int f\chi_b = \int_b f.\end{aligned}$$

(ii)⇒(iii) Clear.

(i)⇒(ii) For $a \in \mathscr{A}$, $b \in \mathscr{B}$ and $c \in \mathscr{C}$,

$$\begin{aligned}\mathbb{E}[\chi_{a\wedge c}\chi_b\,|\mathscr{C}] &= \chi_c\mathbb{E}[\mathbb{E}[\chi_a\chi_b\,|\mathscr{B}]\,|\mathscr{C}] = \chi_c\mathbb{E}[\chi_b\mathbb{E}[\chi_a\,|\mathscr{B}]\,|\mathscr{C}]\\ &= \chi_c\mathbb{E}[\chi_b\mathbb{E}[\chi_a\,|\mathscr{C}]\,|\mathscr{C}] = \mathbb{E}[\chi_{a\wedge c}\,|\mathscr{C}]\,\mathbb{E}[\chi_b\,|\mathscr{C}].\end{aligned}$$

As every $a' \in \mathscr{A}'$ is a disjoint union of events of the form $a \wedge c$, we obtain (5) for $\chi_{a'}$. ■2.8

2.2. CONSTRUCTING A CAT. Starting with the category $\mathfrak{M}$, we look for an underlying logical structure following the method described in [Ben03a, Section 2.3].

The category $\mathfrak{M}$ of probability algebras is a **concrete category** [Ben03a, Definition 2.26], which satisfies in addition:

**Injectiveness**: Since $a = 0 \iff \mu(a) = 0$, measure-preserving morphisms have to be injective.

**Tarski-Vaught property**: This one says that if $\mathscr{A}, \mathscr{B} \leq \mathscr{C}$ and $\mathscr{A} \subseteq \mathscr{B}$ as sets, then $\mathscr{A} \leq \mathscr{B}$. It should be clear from the definitions.

**Elementary chain property**: If $(\mathscr{A}_i\colon i \in I)$ is an increasing chain (according to $\leq$), then $\mathscr{A}_j \leq \bigcup_i \mathscr{A}_i$ for every $j$, and $\bigcup_i \mathscr{A}_i$ is minimal as such.

**Amalgamation**: Any two probability measure algebras can be amalgamated over a common sub-algebra: use free amalgams, for instance.

Thus, the **type** of a tuple [Ben03a, Definition 2.30] is precisely the isomorphism type of the probability algebra it generates. For every set of indices $I$, $\mathrm{S}_I(\mathfrak{M})$ is the set of all $I$-types, i.e., types of $I$-tuples (this is indeed a set and not a proper class). If $f\colon I \to J$ is any mapping, it induces a mapping $f^*\colon \mathrm{S}_J(\mathfrak{M}) \to \mathrm{S}_I(\mathfrak{M})$ sending the type of a $J$-tuple $(a_j\colon j \in J)$ to the type of $(a_{f(i)}\colon i \in I)$. This defines a contravariant functor $\mathrm{S}(\mathfrak{M})$ from sets to sets.

The type of a finite tuple is determined by the probabilities of the atoms of the algebra it generates. Let us convene that $a^0 = a$, $a^1 = a^c$, and if $a_{<n}$ is a finite tuple and $\bar\varepsilon \in 2^n$ then $\bar a^{\bar\varepsilon} = \bigwedge_{i<n} a_i^{\varepsilon_i}$, i.e., the atom corresponding to $\bar\varepsilon$. Then we obtain a bijection between the set of all $n$-types (i.e., types of $n$-tuples) $\mathrm{S}_n(\mathfrak{M})$ and the $(2^n - 1)$-dimensional simplex:

$$\begin{array}{rcl}\mathrm{S}_n(\mathfrak{M}) & \cong & \{\bar w \in [0,1]^{2^n}\colon \sum_{\bar\varepsilon\in 2^n} w_{\bar\varepsilon} = 1\},\\ \mathrm{tp}(a_{<n}) & \mapsto & (\mu(\bar a^{\bar\varepsilon})\colon \bar\varepsilon \in 2^n).\end{array}$$

Let us identify $\mathrm{S}_n(\mathfrak{M})$ with the simplex as above. If $f\colon m \to n$ is any map, then $f^*\colon \mathrm{S}_n(\mathfrak{M}) \to \mathrm{S}_m(\mathfrak{M})$ is given by

$$f^*(\bar{w})_{\bar{\varepsilon}} = \sum_{\bar{\delta}\in 2^n\colon (\delta_{f(0)},\ldots,\delta_{f(m-1)})=\bar{\varepsilon}} w_{\bar{\delta}}.$$

Thus, considering each $\mathrm{S}_n(\mathfrak{M})$ with the topology induced from $\mathbb{R}^{2^n}$, it is clear that every $f^*$ is continuous and therefore closed.

By the characterisation of types above, the type of an infinite tuple is determined by the types of its finite sub-tuples, so $\mathfrak{M}$ has *locality of types* [Ben03a, Definition 2.32]. By [Ben03a, Lemma 2.35], $\mathrm{S}(\mathfrak{M})$ is a *set type-space functor* [Ben03a, Definition 2.18]. With the topological structure induced on $\mathrm{S}_n(\mathfrak{M})$ as real simplexes, $\mathrm{S}_n(\mathfrak{M})$ is a compact topological type-space functor, and so $\mathfrak{M}$ is a compact abstract elementary category.

The locality of types tells us that the functor $\mathrm{S}(\mathfrak{M})$ is determined by its restriction to finite index sets. To be more precise, for every set $I$ we have a natural bijection:

$$\mathrm{S}_I(\mathfrak{M}) \cong \varprojlim_{I_0\subseteq I \text{ finite}} \mathrm{S}_{I_0}(\mathfrak{M}).$$

(Since the family of finite subsets of $I$ with inclusions forms a directed system, its image under $\mathrm{S}(\mathfrak{M})$ forms an inverse system, and there is always a mapping from left to right; by locality of types it is injective, and by [Ben03a, Remark 2.34] it is surjective.) This defines on $\mathrm{S}_I(\mathfrak{M})$ a compact and Hausdorff topology, as the inverse limit of compact and Hausdorff spaces.

For $n < \omega$, $\bar{\varepsilon} \in 2^n$ and $r, s \in \mathbb{Q}$, define an $n$-ary predicate $P_{n,\bar{\varepsilon},r,s}(\bar{x})$; let $\mathcal{L}$ be the set of all such predicates, and let $\Delta$ be the generated positive fragment, i.e., the set of all positive quantifier-free $\mathcal{L}$-formulas. We understand the predicate $P_{n,\bar{\varepsilon},r,s}(\bar{x})$ as saying that $s \leq \mu(\bar{x}^{\bar{\varepsilon}}) \leq r$: this defines a natural interpretation of every probability measure algebra as an $\mathcal{L}$-structure. The type of an $n$-tuple in a probability measure algebra determines which predicates of this form it satisfies, and therefore which $\Delta$-formulas it satisfies: thus every $\varphi \in \Delta$ in $n$ free variables defines a subset $[\varphi] \subseteq \mathrm{S}_n(\mathfrak{M})$. Moreover, it is fairly straightforward to verify that these sets form a basis of closed sets for the topology on $\mathrm{S}_n(\mathfrak{M})$ viewed as a real simplex.

Having defined $\Delta$, a $\Pi(\Delta)$**-sentence** is a sentence of the form $\forall\bar{x}\neg\varphi(\bar{x})$ where $\varphi \in \Delta$. By $\mathrm{Th}_{\Pi(\Delta)}(\mathfrak{M})$ we mean the set of all $\Pi(\Delta)$-sentences true in every probability measure algebra (viewed as an $\mathcal{L}$-structure according to the interpretation given above).

THEOREM 2.9: *Let $T^{\mathfrak{M}} = \mathrm{Th}_{\Pi(\Delta)}(\mathfrak{M})$. Then $T^{\mathfrak{M}}$ is a positive Robinson theory (see [Ben03a, Section 2.1]), and $\mathrm{S}(T^{\mathfrak{M}}) \cong \mathrm{S}(\mathfrak{M})$ with the topology induced from $\mathbb{R}$. In particular, $T^{\mathfrak{M}}$ is Hausdorff (i.e., its type spaces are Hausdorff). Moreover, every e.c. model of $T^{\mathfrak{M}}$ is a probability measure algebra.*

*Proof:* Most of this comes from [Ben03a, Theorem 2.38]. Its statement and proof are actually for a signature containing an $n$-ary predicate for every closed subset of $\mathrm{S}_n(\mathfrak{M})$ (see the proof of [Ben03a, Theorem 2.23] on which it relies); but any signature containing a predicate symbol for each member of a basis of closed sets for $\mathrm{S}_n(\mathfrak{M})$ would do just as well, and our signature $\mathcal{L}$ is such.

Since $\mathrm{S}(T^{\mathfrak{M}}) \cong \mathrm{S}(\mathfrak{M})$ as topological functors, $\mathrm{S}(T^{\mathfrak{M}})$ is Hausdorff, so $T^{\mathfrak{M}}$ is.

The said theorem promises us that every e.c. model of $T^{\mathfrak{M}}$ is a subset of a probability algebra, so we have to show that it is closed for boolean combinations. But $y = x^c$ is defined by $\mu(x \wedge y) = 0 \wedge \mu(x^c \wedge y^c) = 0$, and $z = x \wedge y$ is defined by $\mu(z \wedge x^c) = 0 \wedge \mu(z \wedge y^c) = 0 \wedge \mu(x \wedge y \wedge z^c) = 0$, so any e.c. model is closed for boolean combinations. (By "$\mu(x \wedge y) = 0$" we actually mean the formula $P_{2,(0,0),0,0}(x, y)$, "$\mu(x^c \wedge y^c) = 0$" means $P_{2,(1,1),0,0}(x, y)$, etc. Using this technique we may work as if the boolean operations were function symbols in the language.) $\blacksquare_{2.9}$

Finally, we characterise types:

PROPOSITION 2.10: *Let $\mathscr{B}$ be a complete probability algebra, and $a$ an element. Then $\mathrm{tp}(a/\mathscr{B})$ determines and is determined by the conditional probability $\mathbb{P}[a\,|\mathscr{B}]$. If $A$ is any set of events, then $\mathrm{tp}(A/\mathscr{B})$ (determines and) is determined by the types over $\mathscr{B}$ of every event in the algebra generated by $A$.*

*Proof:* Given our language and the fact that $\mathscr{B}$ is an algebra (rather than just a set) it should be clear that $\mathrm{tp}(a/\mathscr{B})$ determines, and is determined by, the mapping $b \mapsto \mu(a \wedge b)$ for $b \in \mathscr{B}$. But the same can be said of $f = \mathbb{P}[a\,|\mathscr{B}]$, as it is the unique $\mathscr{B}$-measurable function satisfying $\int_b f = \mu(a \wedge b)$ for all $b \in \mathscr{B}$.

Similarly, $\mathrm{tp}(A/\mathscr{B})$ is determined by the mapping $(a, b) \mapsto \mu(a \wedge b)$ where $b \in \mathscr{B}$ and $a$ in the algebra generated by $A$. $\blacksquare_{2.10}$

2.3. ADDITIONAL PROPERTIES. For every $n < \omega$ consider the inclusion $f_n \colon n \hookrightarrow n + 1$, and the corresponding restriction map $f_n^* \colon \mathrm{S}_{n+1}(\mathfrak{M}) \to \mathrm{S}_n(\mathfrak{M})$. Then

$$f_n^* \colon (w_\sigma \colon \sigma \in 2^{n+1}) \mapsto (w_{\tau,0} + w_{\tau,1} \colon \tau \in 2^n),$$

which is clearly an open mapping. Thus, by definition of an open cat [Ben03c, Section 3.4]:

PROPOSITION 2.11: *$\mathfrak{M}$ is open; equivalently, the language we gave above for $\mathfrak{M}$ eliminates not only the existential quantifier, but also the universal one.*

We recall from [Benb] that a **definable metric** on a sort of the universal domain is one such that the properties $d(x,y) \geq r$ and $d(x,y) \leq r$ are type-definable. We further recall that every Hausdorff cat with a countable language admits a definable metric which is unique up to uniform equivalence. In our case, the natural metric $d(a,b) = \mu(a \oplus b)$ is definable, and we only conclude that it is uniformly equivalent to any other definable metric.

We recall from [Benb] that a complete model is a subset $M$ of the universal domain which is complete with respect to one (any) definable metric, and the types over $M$ realised in $M$ are dense in $\mathrm{S}(M)$ (we also recall that in an open cat, it suffices to verify this property for 1-types). The complete model plays the role of elementary sub-models of the universal domain in the first order context.

PROPOSITION 2.12:

(i) *The complete models of $T^{\mathfrak{M}}$ are precisely the complete atomless probability algebras.*

(ii) *The complete models of $T^{\mathfrak{M}}$ are $\omega$-saturated in the classical sense (there is a discussion in [Benb] of a weaker notion of $\omega$-saturation which is sometimes required, but this is not the case here).*

(iii) *$T^{\mathfrak{M}}$ is $\omega$-categorical (i.e., all its complete models which are separable in the metric topology are isomorphic). Moreover, it remains $\omega$-categorical when one adds finitely many elements as constants to the language (unlike for first order theories, the moreover part does not follow in general from $\omega$-categoricity for cats; see [BBH] for an example).*

*Proof:* Clearly, a complete model has to be atomless. Conversely, let $\mathscr{A}$ be a complete atomless algebra. Then for every finite tuple $a_{<n} \in \mathscr{A}$, every 1-type over $a_{<n}$ is realised in $\mathscr{A}$ (i.e., for every choice of $\lambda_{\bar{\varepsilon}} \in [0, \mu(\bar{a}^{\bar{\varepsilon}})]$ for $\bar{\varepsilon} \in 2^n$, there is $b \in \mathscr{A}$ such that $\mu(b \wedge \bar{a}^{\bar{\varepsilon}}) = \lambda_{\bar{\varepsilon}}$ for all $\bar{\varepsilon} \in 2^n$). Since every open set in $\mathrm{S}_1(\mathscr{A})$ contains (the set of extensions of) a complete type over a finite tuple, the types realised in $\mathscr{A}$ are dense in $\mathrm{S}_1(\mathscr{A})$.

This argument shows also that every complete atomless algebra is $\omega$-saturated. It follows that $T^{\mathfrak{M}}$ expanded by finitely many constants is $\omega$-categorical: by standard back-and-forth one constructs a partial isomorphism between two dense sets of separable models, which extends uniquely to an isomorphism. ■2.12

By Maharam's theorem [Fre04, 332B], if $\mathscr{A}$ is a complete model of $T^{\mathfrak{M}}$, i.e., a complete atomless probability algebra, then there exists a partition of $1 \in \mathscr{A}$ into (finitely or) countably many disjoint non-empty events $\{a_i: i < \alpha\}$ (where $\alpha \leq \omega$), and distinct infinite cardinals $\{\kappa_i: i < \alpha\}$, such that for inside each $a_i$, $\mathscr{A}$ is isomorphic to the measure algebra of $\{0,1\}^{\kappa_i}$, shrunk by a factor of $\mu(a_i)$. In other words, if we define a semi-measure $\mu_i(a) = \frac{\mu(a \wedge a_i)}{\mu(a_i)}$ and let $\mathscr{A}_{i,0} = (\mathscr{A}, \mu_i)$, $\mathscr{A}_i = \mathscr{A}_{i,0}/\equiv_0$, then $\mathscr{A}_i$ is isomorphic to the measure algebra of $\{0,1\}^{\kappa_i}$. In this case, the density character of $\mathscr{A}$ (the least cardinality of a dense subset) is $\sup \kappa_i$.

Thus the isomorphism type of $\mathscr{A}$ is determined by a mapping $i \mapsto (\mu(a_i), \kappa_i)$, and in density character $\kappa$ there are at most $(\kappa + 2^\omega)^\omega = \kappa^\omega$ models.

In case $\kappa = \omega$, we must have $\alpha = 1$, $a_0 = 1$ and $\kappa_0 = \omega$, which is an alternative proof that every complete separable atomless probability algebra is isomorphic to that of $\{0,1\}^\omega$, which is in turn isomorphic to that of $[0,1]$.

## 3. Random variables

It should be observed first that the property $y = \bigvee_{i<\omega} x_i$ is not type-definable, since it would require us to know the rate at which $\mu(\bigvee_{i<n} x_i)$ converges to $\mu(y)$. It follows that the property of being (the representation of) a function is not definable, as it includes the requirement that $f_t = \bigvee_{s>t} f_s$.

If we remove this problematic requirement, we are left with the class of all decreasing $\mathbb{Q}^+$-sequences $F_0 = \{(f_t: t \in \mathbb{Q}^+): f_s \leq f_t \text{ for all } s > t\}$. On the one hand, for every $\bar{f} \in F_0$ there is a unique positive function $g$ such that $\{g > t\} \leq f_t \leq \{g \geq t\}$ for all $t \in \mathbb{Q}^+$. On the other, if $\mu(\{f = t\}) > 0$, this does not determine $f_t$ entirely. Define $E_F(\bar{x}, \bar{y}) = \bigwedge_{s>t}[x_t \geq y_s \wedge y_t \geq x_s]$: then $E_F$ is an equivalence relation on $F_0$, and two tuples $\bar{f}$, $\bar{f}'$ represent the same function if and only if $E_F(\bar{f}, \bar{f}')$. It follows that $F = F_0/E_F$ is in a natural bijection with the set $L^0_+$ of all positive functions, and we will identify a function $f$ with the equivalence class $\bar{f}/E_F \in F$ that represents it (sometimes, with some ambiguity, we may identify $f$ with $\bar{f}$).

We recall that in general, a property $p_E(x_E)$ of a hyperimaginary variable $x_E = \bar{x}/E$ is type-definable if and only if the property $p_E(\bar{x}/E)$, as a property of the tuple $\bar{x}$ (i.e., the pull-back of $p_E$ to the home sort), is type-definable. The same rule applies for properties of several variables, possibly in several different hyperimaginary sorts (see [Ben03a, Example 2.16]).

For the case of realisations of $F$, we have a more useful criterion. For every $n$-tuple of positive functions $f^0, \ldots, f^{n-1}$, and Borel set $X \subseteq [0,\infty]^n$, we define the event $\{f^{<n} \in X\}$.

LEMMA 3.1: *For every closed $K \subseteq [0,\infty]^n$ and $r \in \mathbb{R}$, the property $\mu(\{x^{<n} \in K\}) \geq r$ is type-definable (in $F^n$).*

*Proof:* Let $\mathcal{K}_n$ denote the family of all subsets $K \subseteq [0,\infty]^n$ for which the statement holds.

For simplicity, let us first consider the case where $n = 1$ and $K = [a,b]$ for some $a, b \subseteq [0,\infty]$, and let a random variable $f$ be represented by $(f_t\colon t \in \mathbb{Q}^+)$. Then $\{f \in K\} = \{a \leq f \leq b\}$, and $\mu(\{f \in K\}) \geq r$ if and only if

$$\models \exists z \Bigg[ \bigwedge_{t<a, t\in\mathbb{Q}^+} z \leq f_t \Bigg] \wedge \Bigg[ \bigwedge_{t>b, t\in\mathbb{Q}^+} z \leq f_t^c \Bigg] \wedge \mu(z) \geq r,$$

since the maximal event satisfying the conditions on $z$ is precisely $\{a \leq f \leq b\}$. (We remind the reader that an existential quantification of a partial type is logically equivalent to a partial type.)

Now let $n$ be any natural number, and $K$ a union of $m$ closed boxes: $K = \bigcup_{i<m} \prod_{j<n} [a_{ij}, b_{ij}] \subseteq [0,\infty]^n$. Then by the same reasoning, $\mu(\{f^{<n} \in K\}) \geq r$ if and only if, for some (any) representatives $(\bar{f}^j\colon j < n)$, there exist events $(z_{ij}\colon i < m, j < n)$ such that:

(i) For all $i < m$ and $j < n$,

$$\Bigg[ \bigwedge_{t<a_{ij}, t\in\mathbb{Q}^+} z_{ij} \leq f_t^j \Bigg] \wedge \Bigg[ \bigwedge_{t>b_{ij}, t\in\mathbb{Q}^+} z_{ij} \leq (f_t^j)^c \Bigg].$$

(ii) And in addition,

$$\mu\Bigg( \bigvee_{i<m} \bigwedge_{j<n} z_{ij} \Bigg) \geq r.$$

This is a type-definable property of the representatives $(\bar{f}^j\colon j < n)$. Thus $\mu(\{f^{<n} \in K\}) \geq r$ is a type-definable property of $f^{<n}$, and every finite union of closed boxes is in $\mathcal{K}_n$.

Assume now that $K_i \in \mathcal{K}_n$ for every $i < \omega$, $(K_i\colon i < \omega)$ is a decreasing intersection, and $K = \bigcap_i K_i$. Then $\{f^{<n} \in K\} = \bigwedge_{i<\omega} \{f^{<n} \in K_i\}$, and $\mu(\{f^{<n} \in K\}) = \inf_{i<\omega} \{f^{<n} \in K_i\}$. Thus

$$\mu(\{f^{<n} \in K\}) \geq r \iff \bigwedge_{i<\omega} [\mu(\{f^{<n} \in K_i\}) \geq r],$$

and the latter is type-definable by assumption on the $K_i$. As every closed set $K \subseteq [0,\infty]^n$ is the intersection of a countable decreasing sequence of finite unions of closed boxes, we have proved that $\mathcal{K}_n$ contains every closed set. $\blacksquare_{3.1}$

We leave it to the reader to verify that the arithmetic operations we considered above for positive functions are type-definable on $F$ (i.e., their graphs are).

More restricted classes of functions, such as real-valued ones, or $L_p$, are not type-definable, since they do not impose any uniform bound on the distribution. On the other hand, if we are given a uniform bound on the distribution in the form of a positive real-valued function $f$, then the class

$$\begin{aligned} B_f &= \left\{ g\colon \mathscr{A} \to \mathbb{C}\colon \bigwedge_{s>t} \mu(\{|g_s| > s\}) \leq \mu(\{f > t\}) \right\} \\ &= \left\{ g\colon \mathscr{A} \to \mathbb{C}\colon \bigwedge_{s>t} \mu(\{|g_s| \leq s\}) \geq 1 - \mu(\{f > t\}) \right\} \end{aligned}$$

is type-definable by Lemma 3.1 (without parameters: we don't really need $f$, we only need its distribution). Moreover, if $f \in L_p$ then $B_f \subseteq L_p$, and if $f \in L_1$ then integration is definable on $B_f$, by which we mean that for every closed subset $X \subseteq \mathbb{C}$, the set $\{g \in B_f\colon \int g \in X\}$ is type-definable. We leave this to the reader as an exercise, pointing out that the way integration will be defined must depend on the distribution of $f$.

Let us return to the type-definable set $F = L^0_+$ of all positive functions (that is, valued in $[0, \infty]$). We recall from [Benb] that the set $F$ carries a natural topology, namely the logic topology, which is the one induced by any definable metric, as long as one exists. Moreover, since the language is countable, a definable metric exists on every countable hyperimaginary sort, and in particular on that of $F$, and we might as well look for a natural one.

First, for every real number $\varepsilon > 0$ define

$$\delta_\varepsilon = \{(x, y) \in [0, \infty]^2\colon |x - y| \leq \varepsilon \vee x, y \geq 1/\varepsilon\},$$

noting that whatever value we may assign to $|\infty - \infty|$ is irrelevant to the definition of $\delta_\varepsilon$. Moreover, if we interpret $1/0 = \infty$, then $\delta_0$ is also well-defined and is precisely the diagonal of $[0, \infty]^2$. For $\varepsilon > 0$, $\delta_\varepsilon$ is a neighbourhood of this diagonal, and the family $\{\delta_\varepsilon\colon \varepsilon \in (0, 1)\}$ is a basis for its neighbourhoods.

For $f, g \in L^0_+$ define

$$\begin{aligned} &d(f, g) \leq \varepsilon \iff \mu(\{(f, g) \in \delta_\varepsilon\}) \geq 1 - \varepsilon, \\ &d(f, g) = \inf\{\varepsilon \in [0, 1]\colon d(f, g) \leq \varepsilon\}. \end{aligned}$$

Note that the event $\{(f,g) \in \delta_\varepsilon\}$ is by definition equal to

$$\{|f-g| \le \varepsilon\} \vee \{f, g \ge 1/\varepsilon\},$$

and that the infimum is actually attained, by continuity of the measure.

It is fairly straightforward to verify that $d(f,g) \ge \varepsilon$ if and only if

$$\mu(\{|f-g| \ge \varepsilon\} \wedge (\{f \le 1/\varepsilon\} \vee \{g \le 1/\varepsilon\})) \ge \varepsilon.$$

Thus, by Lemma 3.1, both $d(f,g) \le \varepsilon$ and $d(f,g) \ge \varepsilon$ are type-definable properties, and $d$ is a definable metric.

It should be clear from the definition that if $f$ has values in $[0,\infty)$, then $d(g_n, f) \to 0$ if and only if $g_n \to f$ in measure. Since every two definable metrics are uniformly equivalent, this property holds for every definable metric $d$ on $F$.

## 4. Stability

We prove that $\mathfrak{M}$ is stable, and in fact $\omega$-stable, and characterise dividing independence.

4.1. Independence. Let $a, b$ and $c$ be tuples of events, and $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ the boolean algebras generated by $ca$, $cb$ and $c$, respectively. Define $a \mathop{\smile\!\!\!\!\!\!\;|}_c b$ if $\mathscr{A} \mathop{\smile\!\!\!\!\!\!\;|}_{\mathscr{C}} \mathscr{B}$.

Then this relation satisfies:

**Symmetry**: Clear by Lemma 2.7.

**Transitivity**: Clear by Lemma 2.8.

**Extension and stationarity**: By this we mean that given $a$, $b$ and $c$, there is $a'$ such that $\mathrm{tp}(a'/c) = \mathrm{tp}(a/c)$ and $a' \mathop{\smile\!\!\!\!\!\!\;|}_c b$, and that these two properties determine $\mathrm{tp}(a'/bc)$. For this we may replace each of $a$, $b$ and $c$ by $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as above. Let $\mathscr{D} = \mathscr{A} \otimes_{\mathscr{C}} \mathscr{B}$, and embed it in the universal domain sending the copy of $\mathscr{B}$ in $\mathscr{D}$ back to $\mathscr{B}$. Then the copy of $\mathscr{A}$ in $\mathscr{D}$ gets sent to some $\mathscr{A}'$, and $a$ gets sent to some $a'$ satisfying the conditions. Conversely, by the uniqueness of the free amalgam, if $\mathrm{tp}(a''/c) = \mathrm{tp}(a/c)$ and $a'' \mathop{\smile\!\!\!\!\!\!\;|}_c b$, and $\mathscr{A}''$ is generated from $a''c$, then $\mathscr{A}'' \wedge \mathscr{B}$ is isomorphic over $\mathscr{B}$ to $\mathscr{A}' \wedge \mathscr{B}$, whereby $a' \equiv_{bc} a''$.

**Finite character**: Let $A$ be an infinite set of events, and $\mathscr{A}$ generated by $A \cup \mathscr{C}$. If $A \mathop{\smile\!\!\!\!\!\!\;|}_{\mathscr{C}} \mathscr{B}$, then clearly $A_0 \mathop{\smile\!\!\!\!\!\!\;|}_{\mathscr{C}} \mathscr{B}$ for every (finite)

$A_0 \subseteq A$. Conversely, if $a \in \mathscr{A}$, then $a$ is generated by $\mathscr{C}$ and some finite $A_0 \subseteq A$, so it suffices to know that $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathscr{C}} \mathscr{B}$ to conclude that $\mathbb{E}[\chi_a|\mathscr{B}] = \mathbb{E}[\chi_a|\mathscr{C}]$.

**Local character**: Let $\mathscr{A}$ and $\mathscr{B}$ be boolean algebras, and assume at first that $\mathscr{B}$ is complete.

For every $a \in \mathscr{A}$ define $f_a = \mathbb{E}[\chi_a|\mathscr{B}] \in L^\infty(\mathscr{B})$, and let $\mathscr{C} \leq \mathscr{B}$ be the boolean algebra generated by all the events of the form $\{f_a > t\}$ for $t \in \mathbb{Q}^+$ (in other words, $\mathscr{C}$ is the minimal boolean algebra by which all the $f_a$ are measurable). In particular, $\mathbb{E}[\chi_a|\mathscr{B}] = f_a = \mathbb{E}[\chi_a|\mathscr{C}]$ for every $a \in \mathscr{A}$, so $\mathscr{A} \mathop{\smile\!\!\!\!\!\!\mid}_{\mathscr{C}} \mathscr{B}$ by Lemma 2.8, and $|\mathscr{C}| \leq |\mathscr{A}| + \omega$.

If $\mathscr{B}$ is not complete, then every event of the form $\{f_a > t\}$ is in the completion of $\mathscr{B}$, but not necessarily in $\mathscr{B}$ itself. In that case, it is in the completion of a countable sub-algebra, and we can take $\mathscr{C} \leq \mathscr{B}$ to be generated by all of those: then we still have that every $f_a$ is $\mathscr{C}$-measurable and $|\mathscr{C}| \leq |\mathscr{A}| + \omega$, so the same argument goes through.

Which implies:

THEOREM 4.1: *$\mathfrak{M}$ is stable, non-dividing coinciding with the classical notion of independence of events.*

*Moreover, every type over a set of events is stationary: therefore, every set of events is boundedly closed.*

*Proof:* Stationarity implies the independence theorem for types over sets of events, and thus in particular for Lascar strong types. Therefore, by [Ben03b, Theorem 1.51], $\mathfrak{M}$ is simple, with all types being extendible (by the extension property; it is also proved in [Ben03c] that in a simple Hausdorff cat every type is necessarily extendible). Stationarity is also a special case of types having bounded multiplicity, so by [Ben03b, Theorem 2.8], $\mathfrak{M}$ is stable.

For the moreover part, stationarity of types of tuples of events over sets of events implies the stationarity of types of hyperimaginary elements over sets of events. Thus, if $A$ is a set of events and $a \in \mathrm{bdd}(A)$ is possibly hyperimaginary, then every extension of $\mathrm{tp}(a/A)$ to a type over $Aa$ is non-dividing. By stationarity, $\mathrm{tp}(a/Aa)$ is the unique such extension, whereby $a \in \mathrm{dcl}(A)$. Thus $\mathrm{bdd}(A) = \mathrm{dcl}(A)$ and $A$ is boundedly closed. $\blacksquare_{4.1}$

4.2. Entropy.

*Definition 4.2:* Let $a_{<n}$ be a finite set of events, $B$ any set, and let $\mathscr{B}$ be the complete algebra it generates. The **entropy** of $\bar{a}$ over $B$ is defined as

$$H(\bar{a}/B) = -\int \sum_{\bar{\varepsilon}\in 2^n} \mathbb{P}\left[\bar{a}^{\bar{\varepsilon}} \middle| \mathscr{B}\right] \lg_2 \circ \mathbb{P}\left[\bar{a}^{\bar{\varepsilon}} \middle| \mathscr{B}\right],$$

where we extend $x \lg_2 x$ by continuity to $0 \lg_2 0 = 0$. In other words, if $\mathscr{A}$ is the finite algebra generated by $\bar{a}$, then the sum is taken over all the atoms of $\mathscr{A}$.

Fact 4.3: For any finite tuples $\bar{a}$ and $\bar{b}$, and for possibly infinite sets $C$ and $D$, the following hold:

(i) (Monotonicity) $H(\bar{a}/C) \geq H(\bar{a}/CD)$, and equality holds if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\downarrow}_C D$.
(ii) $H(\bar{a}/\bigcup_{i<\omega} C_i) = \lim_{n\to\infty} H(\bar{a}/\bigcup_{i<n} C_i)$.
(iii) The complete algebra generated $C$ contains $\bar{a}$ if and only if $H(\bar{a}/C) = 0$.
(iv) (Additivity) $H(\bar{a}\bar{b}/C) = H(\bar{a}/C\bar{b}) + H(\bar{b}/C)$.
(v) (Symmetry) Let $I(\bar{a};\bar{b}/C) = H(\bar{a}/C) - H(\bar{a}/C\bar{b})$. Then $I(\bar{a};\bar{b}/C) = I(\bar{b};\bar{a}/C)$, and it is equal if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\downarrow}_C \bar{b}$.
(vi) We always have $H(a_{<n}/C) \leq n$, and equality holds if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\downarrow} C$ and all the $a_i$ are independent of probability $\frac{1}{2}$.

*Proof:* Most of this is proved in [Wal82, Chapter 4].

A proof that $H(\bar{a}/CD) = H(\bar{a}/C) \implies \bar{a} \mathop{\smile\!\!\!\!\!\downarrow}_C D$ only appears there where $C = \varnothing$; however, replacing probability with conditional probability one can translate the proofs of [Wal82, Theorems 4.4 and 4.8] to work over the algebra generated by any set $C$.

Symmetry is a consequence of additivity.

For $n = 1$ and $C = \varnothing$, the last item is basic calculus, and the general case follows by additivity and monotonicity. $\blacksquare_{4.3}$

It follows that entropy serves as a real-valued dimension function, and independence is characterised by non-decreasing dimension. The information function $I(\bar{a};\bar{b}/C)$ gives a numerical value to the dependence between $\bar{a}$ and $\bar{b}$ over $C$.

4.3. Miscellanea.

Proposition 4.4: *$\mathfrak{M}$ is $\omega$-stable (as defined in [Benb]).*

*Proof:* We need to prove that for every countable set $A$ there is a separable (with respect to the metric) set $B$ realising every 1-type over $A$. Let $X$ and $Y$, respectively, be two copies of $[0,1]$ equipped with the Lebesgue measure. Let $\mathscr{A} = \mathfrak{B}(X)/\equiv_0$ and $\mathscr{B} = \mathfrak{B}(X \times Y)/\equiv_0$, where $\mathfrak{B}(X)$ means the algebra of Borel sets in $X$. The projection $X \times Y \to X$ induces an embedding $\mathscr{A} \leq \mathscr{B}$. Now, the set $A$ can be embedded in a separable atomless complete algebra, which by $\omega$-categoricity is isomorphic to $\mathscr{A}$. Since $\mathscr{B}$ is separable, it would suffice to prove that it realises every type over $\mathscr{A}$.

Let $a$ be any element in an ambient universal domain. We know that $\mathrm{tp}(a/\mathscr{A})$ is determined by its conditional probability $f = \mathbb{P}[a\,|\mathscr{A}]$. This is a function $f\colon \mathscr{A} \to [0,1]$, which we may identify with a measurable function $f\colon X \to Y$. Let $a' \subseteq X \times Y$ be the area under the graph of $f$. Then $\mathbb{P}[a'\,|\mathscr{A}] = f$ by construction, whereby $a' \equiv_{\mathscr{A}} a$. Since $a' \in \mathscr{B}$, we are done. $\blacksquare_{4.4}$

Proposition 4.5: *Let $\mathscr{A}$ and $\mathscr{B}$ be complete probability measure algebras, and let $\mathscr{C}$ be the minimal complete sub-algebra of $\mathscr{B}$ with respect to which $\mathbb{P}[a\,|\mathscr{B}]$ is measurable for every $a \in \mathscr{A}$.*

*Then $\mathrm{Cb}(\mathscr{A}/\mathscr{B})$ (the canonical base of $lstp(\mathscr{A}/\mathscr{B})$, see [Ben03b, Section 3]) is interdefinable with $\mathscr{C}$.*

*Proof:* Since $\mathscr{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{C}} \mathscr{B}$, it would suffice to prove that $\mathrm{Cb}(\mathscr{A}/\mathscr{C})$ is interdefinable with $\mathscr{C}$. We know that $\mathrm{tp}(\mathscr{A}/\mathscr{C})$ is stationary, and therefore Lascar strong; therefore, $\mathrm{Cb}(\mathscr{A}/\mathscr{C})$ is the class of (an enumeration of) $\mathscr{C}$ modulo a type-definable equivalence relation, and we wish to show that this equivalence relation is equality.

Going back to the construction of canonical bases, we see that it suffices to show that for every $\mathscr{D}$, if $\mathscr{D} \equiv_{\mathscr{A}} \mathscr{C}$ (for some fixed enumeration of $\mathscr{C}$ and $\mathscr{D}$) and $\mathscr{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{C}} \mathscr{D}$ (in which case $\mathscr{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathscr{D}} \mathscr{C}$ as well, by comparison of ranks), then $\mathscr{C} = \mathscr{D}$ with the same enumeration. Indeed, in this case we have $\mathbb{P}[a\,|\mathscr{C}] = \mathbb{P}[a\,|\mathscr{C} \wedge \mathscr{D}] = \mathbb{P}[a\,|\mathscr{D}]$ for every $a \in \mathscr{A}$. Since $\mathscr{C} \equiv_{\mathscr{A}} \mathscr{D}$, and $\mathrm{tp}(\mathscr{A}, \mathscr{C})$ says that $\mathscr{C}$ is a complete boolean algebra generated by events of the form $\{\mathbb{P}[a\,|\mathscr{C}] > t\}$, we conclude that $\mathscr{C} = \mathscr{D}$. $\blacksquare_{4.5}$

Recall that if $\langle G, \cdot\rangle$ is a type-definable group (i.e., its domain, as well as the graph of the group operation, are type-definable by partial types) then $g \in G$ is **generic** if for every $a \in G$, $a \mathop{\smile\!\!\!\!\!\!|} g \Longrightarrow g \cdot a \mathop{\smile\!\!\!\!\!\!|} a$. Recall also from [Ben03c] that in a simple thick cat (and therefore, in a stable Hausdorff cat) type-definable groups have generic types. A stable type-definable group is **connected** if it has no type-definable sub-groups of bounded index, or equivalently if it has a unique generic type (this equivalence is proved as for first order theories).

FACT 4.6: $\mathfrak{M}$ interprets a group, namely the additive group of the underlying boolean ring. This group is connected, and its (unique) generic type is that of the event of probability $\frac{1}{2}$.

*Proof:* We know that generic elements exist. Let $a$ be generic, and write $x = \mu(a)$. Assume that $a \perp b$, and write also $y = \mu(b)$, assuming it is neither 0 nor 1. Then $a \oplus b \perp b$, whereby

$$\begin{aligned}
\mu(a \wedge b) &= \mu(a)\mu(b) = xy, \\
\mu(a \oplus b) &= \mu(a) + \mu(b) - 2\mu(a \wedge b) = x + y - 2xy, \\
y - xy &= \mu(b) - \mu(a \wedge b) = \mu(b \smallsetminus a) \\
&= \mu((a \oplus b) \wedge b) = \mu(a \oplus b)\mu(b) = y(x + y - 2xy) \\
&\Longrightarrow x(2y^2 - 2y) = y^2 - y \\
&\Longrightarrow x = \frac{1}{2}.
\end{aligned}$$

By Proposition 2.10, the fact that $\mu(a) = \frac{1}{2}$ determines $\operatorname{tp}(a)$. As there is a unique generic, the group is connected.

It follows that $a$ is generic over $\mathscr{A}$ if and only if $\mathbb{E}[\chi_a | \mathscr{A}]$ is the constant $\frac{1}{2}$. $\blacksquare_{4.6}$

This could be viewed as a re-statement of the common wisdom saying that if you have absolutely no information on the probability of a given event, just call it fifty-fifty....

## 5. Hyperimaginaries and Galois theory

We show that $T^{\mathfrak{M}}$ admits weak elimination of hyperimaginaries, but there are hyperimaginaries which cannot be eliminated even in favour of finitary ones.

In [BB04a] it was suggested that hyperimaginaries should be understood in two steps: first, understand boundedly-closed hyperimaginaries (i.e., ones for which $\operatorname{bdd}(a) = \operatorname{dcl}(a)$); then understand arbitrary hyperimaginaries via their Galois groups $\operatorname{Gal}(a) = \operatorname{Aut}(\operatorname{bdd}(a)/a)$. Even though [BB04a] was written with the extremely easy case of Hilbert spaces in mind, much of the development holds in arbitrary Hausdorff cats (which are, we recall, cats whose type-spaces are Hausdorff).

### 5.1. Boundedly closed sets.

*Definition 1.1:* A cat $T$ admits **weak elimination of hyperimaginaries** if for every hyperimaginary $a$ there exists a tuple $b \in \operatorname{bdd}(a)$ in the home sort (a "real" tuple) such that $a \in \mathrm{dcl}(b)$; equivalently, if every boundedly closed hyperimaginary is interdefinable with a real tuple.

The following result is proved implicitly in [BB04a] in the special case of Hilbert spaces.

PROPOSITION 5.2: *Let $T$ be a stable Hausdorff cat, and assume that for every two tuples of real elements $a$ and $b$, if $\mathrm{tp}(a/b)$ is stationary then $\mathrm{Cb}(a/b)$ is interdefinable with a real tuple. Then $T$ admits weak elimination of hyperimaginaries.*

*Proof:* Let $a_E$ be the class of a real tuple $a$ modulo a type-definable equivalence relation $E$, and assume that $a_E$ is boundedly closed. Then $\mathrm{tp}(a/a_E)$ is stationary, and it is an easy exercise to see that $\mathrm{Cb}(a/a_E) = a_E$. Let $b \equiv_{a_E} a$ be such that $a \mathop{\smile\!\!\!\!\!\mid}_{a_E} b$. Then $\mathrm{tp}(a/b)$ is stationary, and $\mathrm{Cb}(a/b) = a_E$, which is therefore interdefinable with a real tuple. ∎$_{5.2}$

COROLLARY 5.3: *Every boundedly closed set is interdefinable with a (unique) complete probability algebra.*

*Therefore, $T^{\mathfrak{M}}$ admits weak elimination of hyperimaginaries.*

*Proof:* Assume that $\mathrm{bdd}(a) = \mathrm{dcl}(a)$. Then by Proposition 4.5 and Proposition 5.2, $a$ is interdefinable with a set $A$ of real elements. Thus $a$ is interdefinable with the set of all real elements definable over $A$, which is simply the complete probability algebra generated by $A$.

Clearly, two distinct complete boolean algebras cannot be interdefinable. ∎$_{5.3}$

The same reasoning yielded the same result for Hilbert space in [BB04b].

5.2. GALOIS THEORY. For the time being, we work in an arbitrary Hausdorff cat $T$.

For a hyperimaginary $a$, we define $\mathrm{Aut}(a)$ as the group of all permutations of $\mathrm{dcl}(a)$ induced by automorphisms of the universe. $\mathrm{Aut}(a)$ is an invariant of the interdefinability class of $a$; we would like to render it a topological group.

Recall from [Benb] that the topology induced by the definable metrics (on a fixed sort) is the one for which the type-definable sets (with parameters anywhere in the universal domain) form a basis of closed sets. We call this the **logic topology**. A nice feature of the logic topology is that if $A$ is some subset of the universe (in a single sort), then the type-definable subsets of $A$ with parameters in $A$ form a basis for the induced topology on $A$. Also, if $\bar{x} = (x_i \colon i < \alpha)$ is a tuple of variables in various sorts, then the logic topology on the sort of $\bar{x}$ is the Tychonoff product of the logic topologies on the sorts of the $x_i$ (if $\alpha$ is countable, this follows from the fact that we can combine definable metrics on the sorts of the variables $x_i$ to a definable metric on the sort of $\bar{x}$, which induces the product topology; if $\alpha$ is not countable then essentially the same argument can still be carried through using the notion of an abstract distance defined in [Benb]).

We define the topology on $\mathrm{Aut}(a)$ as that of pointwise convergence, i.e., as a subset of $\mathrm{dcl}(a)^{\mathrm{dcl}(a)}$ in the power topology: this is clearly interdefinability invariant. Note that $g \in \mathrm{Aut}(a)$ is completely determined by $g(a)$, and for every other $b \in \mathrm{dcl}(a)$, $p(x,y) = \mathrm{tp}(a,b)$ defines the graph of a partial mapping which is continuous, sends $a$ to $b$, and commutes with every $g \in \mathrm{Aut}(a)$. It follows that the topology on $\mathrm{Aut}(a)$ is that of convergence on $a$, i.e., the minimal such that the mapping $g \mapsto g(a)$ is continuous. Thus, the family of all subsets of the form $\{g \in \mathrm{Aut}(a) \colon \vDash \varphi(g(a),b)\}$, for any $\varphi(x,b)$, forms a basis of closed sets for $\mathrm{Aut}(a)$. Since moreover $g(a) \in \mathrm{dcl}(a)$ for all such $g$, if we restrict this to the family of all sets of the form $\{g \in \mathrm{Aut}(a) \colon \vDash \varphi(g(a),a)\}$, where $\varphi(x,y)$ is any formula both of whose variables are in the sort of $a$, we still have a basis of closed sets.

As a side remark, assume that $a$ is an enumeration of a set $B = \{b_i \colon i < \lambda\}$, and every $g \in \mathrm{Aut}(a)$ induces a permutation on $B$: then the topology on $\mathrm{Aut}(a)$ is that of pointwise convergence of functions from $B$ to itself. In the case studied in [BB04a], when $a$ is an enumeration of a Hilbert space, $\mathrm{Aut}(a)$ is its unitary group equipped with the strong operator topology.

For $g \in \mathrm{Aut}(a)$, define $p^g \in \mathrm{S}_{|a|\times 2}(T)$ as $\mathrm{tp}(g(a),a)$. Then the preceding discussion proves:

Proposition 5.4: *The mapping $g \mapsto p^g$ is a topological embedding of* $\mathrm{Aut}(a)$ *in* $\mathrm{S}_{|a|\times 2}(T)$.

It follows that:

Lemma 5.5: *The map $m_0$: $\mathrm{Aut}(a)^2 \to \mathrm{S}_{|a|\times 3}(T)$ defined by*

$$(g,h) \mapsto \mathrm{tp}(gh(a),h(a),a)$$

*is a topological embedding.*

*Proof:* The map is clearly injective, and we need to show that it is a homeomorphism with its image.

The important part is the continuity. Fix a pair $(g,h) \in \mathrm{Aut}(a)^2$, and let $p(x,y,z) = m_0(g,h) = \mathrm{tp}(gh(a),h(a),a)$. Consider a basic neighbourhood of $p$, defined by a negative formula $\neg\varphi(x,y,z)$. Then $p^g(x,y) \wedge p^h(y,z) \wedge \varphi(x,y,z)$ is contradictory, and since $T$ is Hausdorff there are neighbourhoods $\neg\psi$ of $p^g$, and $\neg\chi$ of $p^h$ such that $\neg\psi(x,y) \wedge \neg\chi(y,z) \vdash \neg\varphi(x,y,z)$. Let $U$ and $V$ be the pull-backs of $\neg\psi(x,y)$ and $\neg\chi(x,y)$ from $\mathrm{S}_{|a|\times 2}(T)$ to $\mathrm{Aut}(a)$. Then $U \times V$ is a neighbourhood of $(g,h)$ whose image under $m_0$ is contained in $\neg\varphi(x,y,z)$. Thus $m_0$ is continuous at every point $(g,h) \in \mathrm{Aut}(a)^2$.

To see that it is actually an embedding, we observe that a basic closed set in $\mathrm{Aut}(a)^2$ is of the form $\varphi(x,y) \times \psi(x,y)$, where we identify $\mathrm{Aut}(a)$ with its image in $\mathrm{S}_{|a|\times 2}(T)$ and a formula with the set of its realisations in the image. The image of $\varphi(x,y) \times \psi(x,y)$ is defined (in the image of $m_0$) by $\varphi(x,y) \wedge \psi(y,z)$, which is closed as required. $\blacksquare_{5.5}$

And we conclude:

PROPOSITION 5.6: *With the topology defined as above,* $\mathrm{Aut}(a)$ *is a topological group.*

*Proof:* Inverse is clearly continuous. To see that product is continuous we use Lemma 5.5 and the fact that $p^{gh}(x,y) = \exists z p^g(x,z) \wedge p^h(z,y)$, which defines a continuous map $\mathrm{S}_{|a|\times 3}(T) \to \mathrm{S}_{|a|\times 2}(T)$. $\blacksquare_{5.6}$

This topology is not, in general, compact. However, the compact subgroups correspond to hyperimaginaries:

THEOREM 5.7: *There is a Galois correspondence between compact subgroups of* $\mathrm{Aut}(a)$ *and interdefinability classes of hyperimaginaries* $b$ *satisfying:*

(i) $b \in \mathrm{dcl}(a) \subseteq \mathrm{bdd}(b)$.

(ii) *Every automorphism fixing* $b$ *(pointwise) fixes* $\mathrm{dcl}(a)$ *setwise.*

*(The first condition may be interpreted as saying that* $a$ *is a bounded extension of* $b$*, and the second as saying that it is a Galois extension.)*

*The correspondence associates to each such* $b$ *the subgroup*

$$\mathrm{Aut}(a/b) = \{f \in \mathrm{Aut}(a) \colon f(b) = b\}.$$

*Proof:* Assume that $b$ satisfies the conditions. If $p(x,t) = \mathrm{tp}(a,b)$, then $\mathrm{Aut}(a/b)$ is defined in $\mathrm{S}_{|a|\times 2}(T)$ by $\exists t p(x,t) \wedge p(y,t)$. It is therefore compact.

Given a compact subgroup $G \leq \mathrm{Aut}(a)$, it is defined in $\mathrm{S}_{|a|\times 2}(T)$ by a partial type $G(x,y)$. Then $G(x,y)$ is an equivalence relation on $\mathrm{tp}(a)$, and we can define $b = a/G$. Then $b \in \mathrm{dcl}(a)$ by definition, and the set of $b$-conjugates of $a$ is $\{g(a): g \in G\}$, which is bounded, whereby $a \subseteq \mathrm{bdd}(b)$.

Finally, we show the correspondence. First, let $b$ satisfy the conditions $G = \mathrm{Aut}(a/b)$ and $c = a/G$. Then an automorphism fixing $b$ fixes $\mathrm{dcl}(a)$ setwise by assumption, and acts on $a$ as an element of $G$, whereby it fixes $c$. Conversely, if an automorphism fixes $c$, it must send $a$ to $g(a)$ for some $g \in G$: since $b \in \mathrm{dcl}(a)$ and $g \in \mathrm{Aut}(a/b)$, $b$ must be fixed. Thus $b$ and $c$ are interdefinable as required.

In the other direction, let $G \leq \mathrm{Aut}(a)$ be compact and $b = a/G$. Then one verifies easily that $G = \mathrm{Aut}(a/b)$. $\blacksquare_{5.7}$

*Remark 5.8:* As usual, if $c \subseteq a$ is a Galois extension and $c \subseteq b \subseteq a$ then $b \subseteq a$ is always a Galois extension, and $c \subseteq b$ is a Galois extension if and only if $\mathrm{Aut}(a/b) \triangleleft \mathrm{Aut}(a/c)$.

For every hyperimaginary $a$, define the **(absolute) Galois group** of $a$ as

$$\mathrm{Gal}(a) = \mathrm{Aut}(\mathrm{bdd}(a)/a).$$

COROLLARY 5.9: *For every hyperimaginary $a$, $\mathrm{Gal}(a)$ is a compact subgroup of $\mathrm{Aut}(\mathrm{bdd}(a))$, and $a$ is interdefinable with $\mathrm{bdd}(a)/\mathrm{Gal}(a)$. Conversely, if $a$ is boundedly closed, $G \leq \mathrm{Aut}(a)$ is compact, and $b = a/G$, then $\mathrm{Gal}(b) = G$.*

(This was proved in the particular case of Hilbert spaces in [BB04a], using more concrete means.)

5.3. THE CASE OF PROBABILITY ALGEBRAS. Previous results yield that every hyperimaginary is interdefinable with $\mathscr{A}/G$, where $\mathscr{A}$ is a complete probability algebra, and $G \leq \mathrm{Aut}(\mathscr{A})$ is compact in the topology of pointwise convergence. Also, since the language is countable, every hyperimaginary is interdefinable with a sequence of countable hyperimaginaries (i.e., quotients of countable tuples).

We conclude by showing that this is indeed the best we can do: unlike Hilbert spaces, we cannot eliminate hyperimaginaries to finitary ones.

Indeed, let $a$ be a finitary hyperimaginary, say $b_{<n}/E$ where $b_{<n}$ is a finite tuple of events and $E$ an equivalence relation. Then $\mathrm{bdd}(a)$ is interdefinable with a complete algebra $\mathscr{A}$, and $\mathscr{A} \subseteq \mathrm{bdd}(b_{<n})$. Then in fact $\mathscr{A}$ is a subalgebra of the one generated by $b_{<n}$, which is finite. It follows that $\mathrm{Aut}(\mathscr{A})$ is finite, and therefore so is $\mathrm{Gal}(a) = \mathrm{Aut}(\mathscr{A}/a)$.

Now let $a$ be any hyperimaginary which is interdefinable with a tuple of finitary ones, say $a_{<\alpha}$. For $i < \alpha$ let $\mathscr{A}_i = \mathrm{bdd}(a_i)$ be finite algebras as above.

Then $\mathrm{bdd}(a) \subseteq \mathrm{bdd}(\bigcup_i \mathscr{A}_i)$. But since $\bigcup_i \mathscr{A}_i$ is a set of real elements and therefore boundedly closed, we see that $\mathrm{bdd}(a) = \mathscr{A}$, the complete algebra generated by $\bigcup_i \mathscr{A}_i$. Then every $g \in \mathrm{Aut}(\mathscr{A})$ fixing $a_{<\alpha}$ pointwise restricts to an automorphism of each $\mathscr{A}_i$, and is determined by the restrictions $g_i = g \restriction_{\mathscr{A}_i} \in \mathrm{Gal}(a_i)$. Therefore $\mathrm{Gal}(a) = \varprojlim_{w \in [\alpha]^{<\omega}} \mathrm{Gal}(a_{\in w})$ is a pro-finite group.

On the other hand, let $G$ be any compact group, and $\mu$ its Haar measure. (The following must be well-known, but I somehow failed to find a textbook where it is done.) The Haar measure on $G$ can be constructed via a positive integration functional on continuous complex-valued function on $G$. By the Riesz representation theorem [Rud66, Theorem 2.14] it is regular, i.e., for every Borel set $A$,

$$\mu(A) = \sup\{\mu(F) \colon F \subseteq A \text{ closed}\} = \inf\{\mu(U) \colon A \subseteq U \text{ open}\}.$$

Thus for every $\varepsilon > 0$ there are a closed and an open set $F \subseteq A \subseteq U$ such that $\mu(U) - \mu(F) < \varepsilon$, and a neighbourhood $V$ of the identity such that $V \cdot F \cup V^{-1} \cdot F \subseteq U$. If follows that for every $g \in V$,

$$\mu(A \oplus gA) \leq \mu(U \setminus gF) + \mu(gU \setminus F) < 2\varepsilon.$$

This shows $G$ acts continuously on every event in $\mathscr{A}$, and since the topology $\mathrm{Aut}(\mathscr{A})$ is that of pointwise convergence, the mapping $G \hookrightarrow \mathrm{Aut}(\mathscr{A})$ is continuous. Since $G$ is compact and $\mathrm{Aut}(\mathscr{A})$ Hausdorff, it is in fact an embedding, and $G$ can be viewed as a compact sub-group of $\mathrm{Aut}(\mathscr{A})$. By Theorem 5.7, there is a hyperimaginary $a = \mathscr{A}/G$, and $\mathrm{Gal}(a) = G$.

Since $G$ is not necessarily pro-finite, we see there are hyperimaginaries which are not interdefinable with a tuple of finitary ones.

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